# Linear Algebra for Computer Science

## An incremental document:
## From Systems and Matrices to Eigenvalues and Eigenvectors

Francisco Escolano

**Notes based on the text**
**“Elementary Linear Algebra” by Larson and Falvo, 6th edition**

# Exercise

**Time and equations**

When the LINEAR COMBINATIONS OF UNKNOWNS are referred TO TIME COORDINATES

**Now, the linear combination of $Ax_1+Bx_2+Cx_3$ is unknown. However, 1 day ago it was decreased by 600 but modifying $x_1+1.5$, $x_2-0.5$ and $x_3+1$. Finally, 2 days ago, the linear combination increased in 350 if we had $(x_1+1.5)-1$, $(x_2-0.5)-1.5$ and $(x_3+1)+0.5$.**

**Show that: a) $x_1$, $x_2$, $x_3$ cannot be determined with the above information even when we know A, B and C, b) A,B and C, when considered as unknowns, have infinite number of solutions, c) However, if we set $C=200$, then A and B can be determined.**

OPTIONAL: SOLVE THIS SYSTEM WITH SymPy. Before solving, apply the rref function and show the Reduced Echelon form and the pivots

# Exercise-Solution

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When the LINEAR COMBINATIONS OF UNKNOWNS are referred TO TIME COORDINATES

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$$\frac{3}{2}A-\frac{1}{2}B+C=600$$

$$\frac{1}{2}A-2B+\frac{3}{2}C=-350$$

$$Ax_1+Bx_2+Bx_3$$

$$A(x_1+1.5)+B(x_2-0.5)+C(x_3+1)=Ax_1+Bx_2+Cx_3+600$$

$$A(x_1+1.5-1)+B(x_2-0.5-1.5)+C(x_3+1+0.5)=Ax_1+Bx_2+Cx_3-350$$

# Linearity vs non-linearity, examples of applications, Gauss and Gauss-Jordan reduction, Echelon form, Homogeneous systems :

```
def example_rref():
    from sympy.interactive.printing import init_printing
    init_printing(use_unicode=False, wrap_line=False)
    from sympy.matrices import Matrix
    Ab= Matrix([[1,1,0,0,0,300],[0,-1,-1,0,-1,200],[1,0,1,1,0,150],[0,0,0,1,1,350]])
    r,pivots = Ab.rref()
    return r,pivots
```

# 2. Vectors and matrices

## Dot product vs matrix product: systems, Properties of matrices operations, Inverse and transpose, Elementary matrices and inverses, Applications to graphs:

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# Dot product vs matrix product: systems, Properties of matrices operations, Inverse and transpose, Elementary matrices and inverses, Applications to graphs:

Linear Systems contain row
And column vectors

$$A_{m\times n}\mathbf{x}_{n\times 1} = \mathbf{b}_{m\times 1}$$

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Notation for rows and columns

Row

$$\mathbf{a_{i:}} = [a_{i1} \quad a_{i2} \quad \dots \quad a_{in}] = \begin{bmatrix} a_{i1} \\ a_{i2} \\ \vdots \\ a_{in} \end{bmatrix}^T$$

Col

$$\mathbf{a_{:j}} = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix} = [a_{1j} \quad a_{2j} \quad \dots \quad a_{mj}]^T$$

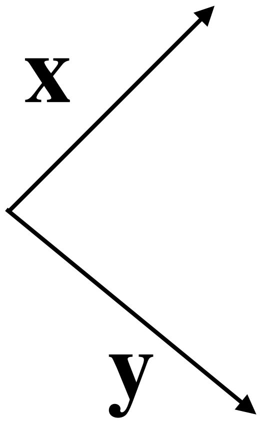
Dot product (is a scalar)

$$\mathbf{x}_{k\times 1}, \mathbf{y}_{k\times 1} \in \mathbb{R}^k$$

$$<\mathbf{x}, \mathbf{y}> = \mathbf{x}^T\mathbf{y} = [x_1 \quad x_2 \quad \dots x_k] \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_k \end{bmatrix} = \sum_{j=1}^{k} x_j y_j \in \mathbb{R}$$

$$\mathbf{x} = [3\ 1\ 5]^T, \mathbf{y} = [-1\ 3\ 0]^T$$

$$<\mathbf{x}, \mathbf{y}> = \sum_{j=1}^{k} x_j y_j = 3 \cdot (-1) + 1 \cdot 3 + 5 \times 0 = 0$$



Dot product quantifies
Dependence/correlation

# Dot product vs matrix product: systems, Properties of matrices operations, Inverse and transpose, Elementary matrices and inverses, Applications to graphs:

Linear Systems can be explained as dot products (row view)

$$A_{m\times n}\mathbf{x}_{n\times 1} = \mathbf{b}_{m\times 1}$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Solving a Linear system can be seen as finding the unknowns that get the m proper dot products

$$\begin{bmatrix} \mathbf{a}_{1:} \\ \mathbf{a}_{2:} \\ \vdots \\ \mathbf{a}_{m:} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \Rightarrow \begin{bmatrix} < \mathbf{a}_{1:}, \mathbf{x} > \\ < \mathbf{a}_{2:}, \mathbf{x} > \\ \vdots \\ < \mathbf{a}_{m:}, \mathbf{x} > \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \Rightarrow \begin{bmatrix} \sum_{j=1}^{n} a_{1j}x_j \\ \sum_{j=1}^{n} a_{2j}x_j \\ \vdots \\ \sum_{j=1}^{n} a_{mj}x_j \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

**This is consistent with seeing each of the m $b_j$ as the linear combination of the n unknowns $x_i$ where the coefficients are that of the j-th row $\mathbf{a}_{j:}$**

# Dot product vs matrix product: systems, Properties of matrices operations, Inverse and transpose, Elementary matrices and inverses, Applications to graphs:

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**Matrix product (is a matrix)**

$$\mathbf{A}_{m\times k}, \mathbf{B}_{k\times n} \Rightarrow \mathbf{C} = \mathbf{AB} \in \mathbb{R}^{m\times n}$$

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1k} \\ a_{21} & a_{22} & \dots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ik} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mk} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1j} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2j} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ b_{k1} & b_{k2} & \dots & b_{kj} & \dots & b_{kn} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1j} & \dots & b_{1n} \\ c_{21} & c_{22} & \dots & c_{2j} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{i1} & c_{i2} & \dots & c_{ij} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \dots & c_{mj} & \dots & c_{mn} \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{a}_{1:} \\ \mathbf{a}_{2:} \\ \vdots \\ \mathbf{a}_{m:} \end{bmatrix} \begin{bmatrix} \mathbf{b}_{:1} & \mathbf{b}_{:2} & \dots & \mathbf{b}_{:n} \end{bmatrix} = \begin{bmatrix} < \mathbf{a}_{1:}, \mathbf{b}_{:1} > & < \mathbf{a}_{1:}, \mathbf{b}_{:2} > & \dots & < \mathbf{a}_{1:}, \mathbf{b}_{:n} > \\ < \mathbf{a}_{2:}, \mathbf{b}_{:1} > & < \mathbf{a}_{2:}, \mathbf{b}_{:2} > & \dots & < \mathbf{a}_{2:}, \mathbf{b}_{:n} > \\ \vdots & \vdots & \ddots & \vdots \\ < \mathbf{a}_{m:}, \mathbf{b}_{:1} > & < \mathbf{a}_{m:}, \mathbf{b}_{:2} > & \dots & < \mathbf{a}_{m:}, \mathbf{b}_{:n} > \end{bmatrix}$$

**A product of matrices consists of $m \times n$ dot products between all the rows of A and all the columns of B. This is why their dimensions $k$ must match!**

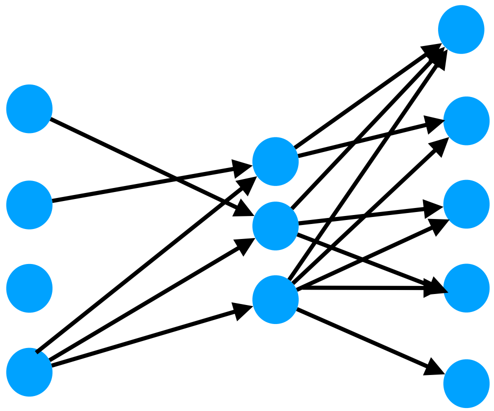
**Example of Matrix product**

## Propagation of a rumor

**Each matrix A, B to multiply relates 2 groups of contact. What is the meaning of AB?**

**Let A be a binary $m \times k$ matrix where each row (1st group of persons) denotes a person sending a rumor and each column denotes a person believing the rumor (2nd group). Similarly, let B be a $k \times n$ binary matrix where each row corresponds to the 2nd group and each column corresponds to a 3rd group. Determine what people in the 3rd group receives finally the rumor started by the first group and many rumors they receive**

$$
\begin{array}{cc} & \text{2nd} \\ \text{1st} & \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \end{array}
\begin{array}{cc} & \text{3rd} \\ \text{2nd} & \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}
= \begin{array}{cc} & \text{3rd} \\ \text{1st} & \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 3 & 2 & 2 & 1 & 2 \end{bmatrix} \end{array}
$$

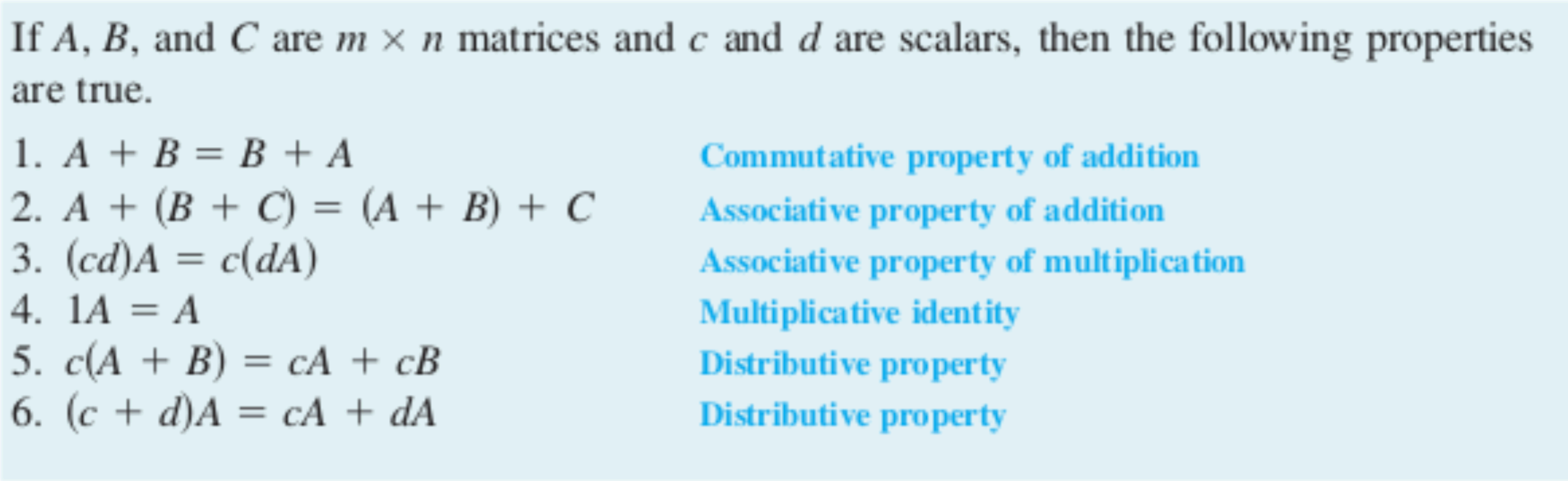


Last person in 1st group and last row in the second group lead to the larger number of contacts. 3rd person in the first group does not send any information at all!

# Dot product vs matrix product: systems, Properties of matrices operations, Inverse and transpose, Elementary matrices and inverses, Applications to graphs:

## Matrix operations

1. **Sum of matrices** $C_{m\times n} = A_{m\times n} + B_{m\times n} \Rightarrow \left[c_{ij}\right] = \left[a_{ij}\right] + \left[b_{ij}\right]$
2. **Product of matrices** $C_{m\times n} = A_{m\times k}B_{k\times n} \Rightarrow \left[c_{ij}\right] = \left[\sum_{r=1}^{k} a_{ir}b_{rj}\right]$
3. **Product of scalar and matrix** $C_{m\times n} = cA_{m\times n} \Rightarrow \left[c_{ij}\right] = \left[c\cdot a_{ij}\right]$



If $A$, $B$, and $C$ are $m \times n$ matrices and $c$ and $d$ are scalars, then the following properties are true.

| | |
|---|---|
| 1. $A + B = B + A$ | Commutative property of addition |
| 2. $A + (B + C) = (A + B) + C$ | Associative property of addition |
| 3. $(cd)A = c(dA)$ | Associative property of multiplication |
| 4. $1A = A$ | Multiplicative identity |
| 5. $c(A + B) = cA + cB$ | Distributive property |
| 6. $(c + d)A = cA + dA$ | Distributive property |

# Dot product vs matrix product: systems, Properties of matrices operations, Inverse and transpose, Elementary matrices and inverses, Applications to graphs:

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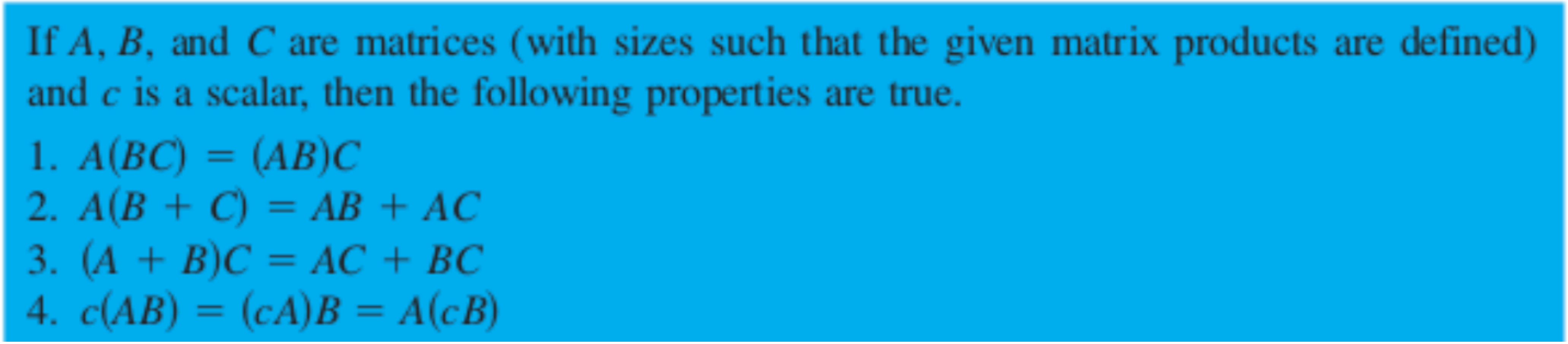
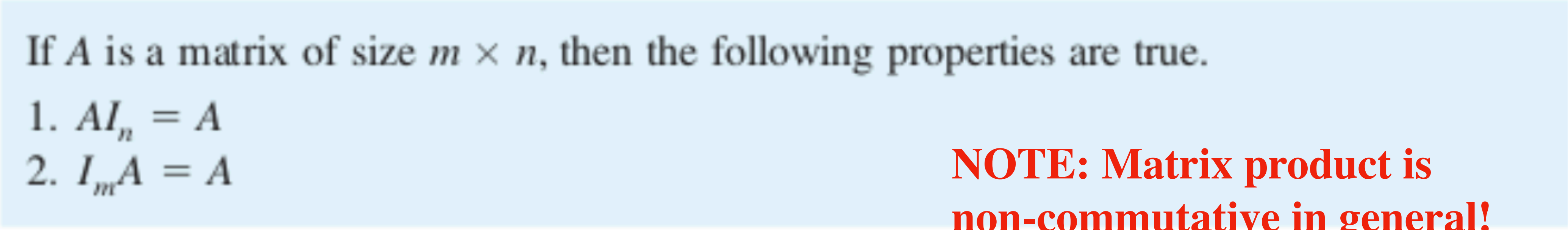


If $A$, $B$, and $C$ are matrices (with sizes such that the given matrix products are defined) and $c$ is a scalar, then the following properties are true.

1. $A(BC) = (AB)C$
2. $A(B + C) = AB + AC$
3. $(A + B)C = AC + BC$
4. $c(AB) = (cA)B = A(cB)$



If $A$ is a matrix of size $m \times n$, then the following properties are true.

1. $AI_n = A$
2. $I_mA = A$

**NOTE: Matrix product is non-commutative in general!**

# Dot product vs matrix product: systems, Properties of matrices operations, Inverse and transpose, Elementary matrices and inverses, Applications to graphs:

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**Matrix Transpose**

$$\mathbf{A}_{m\times k} \Rightarrow C = \mathbf{A}^T_{k\times m} \quad [c_{ij}] = [a_{ji}]$$

**Symmetric Matrices**

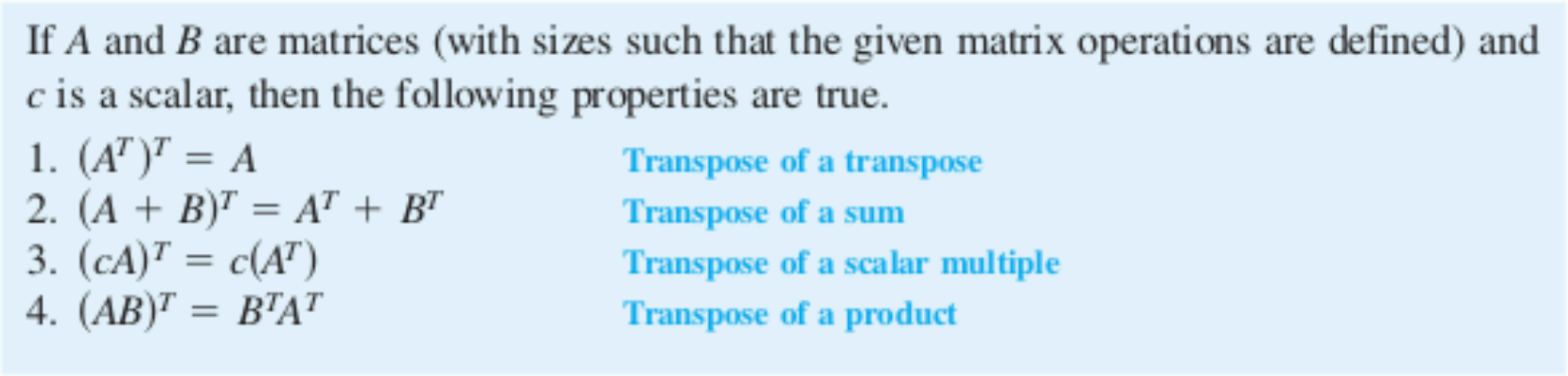
$$\mathbf{A}_{n\times n} = C = \mathbf{A}^T_{n\times n} \quad [c_{ij}] = [a_{ij}]$$

## Matrix transposition

1. Matrix transposition is very useful for creating square matrices from non-square ones

$$C_{m\times m} = A_{m\times n}A^T_{n\times m}$$

2. Since we use the matrix product, the elements of the new matrix consist in the correlations (dot product) of the rows and cols of the matrix A, and **C symmetric!**



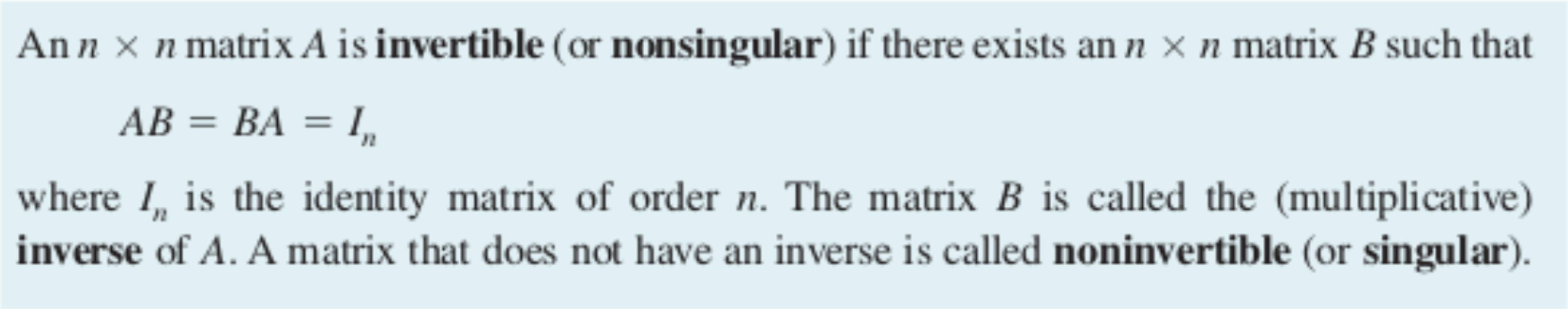
If $A$ and $B$ are matrices (with sizes such that the given matrix operations are defined) and $c$ is a scalar, then the following properties are true.

| | |
|---|---|
| 1. $(A^T)^T = A$ | Transpose of a transpose |
| 2. $(A + B)^T = A^T + B^T$ | Transpose of a sum |
| 3. $(cA)^T = c(A^T)$ | Transpose of a scalar multiple |
| 4. $(AB)^T = B^TA^T$ | Transpose of a product |

# Dot product vs matrix product: systems, Properties of matrices operations, Inverse and transpose, Elementary matrices and inverses, Applications to graphs:

## Matrix Inverse

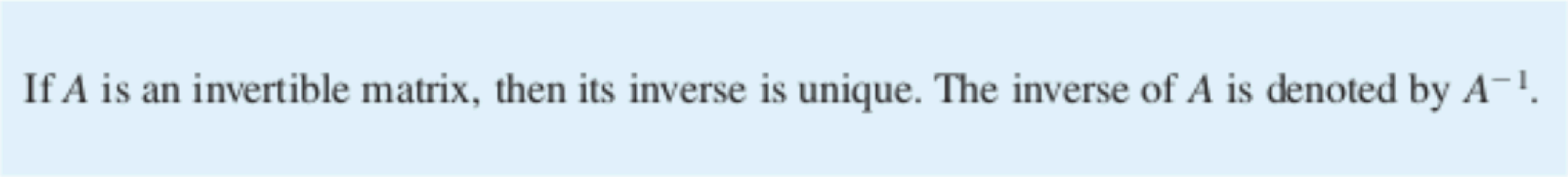
$$\mathbf{A}_{n\times n} \Rightarrow C = \mathbf{A}^{-1}_{n\times n}\ , \exists \mathbf{A}^{-1} : \mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}_n$$



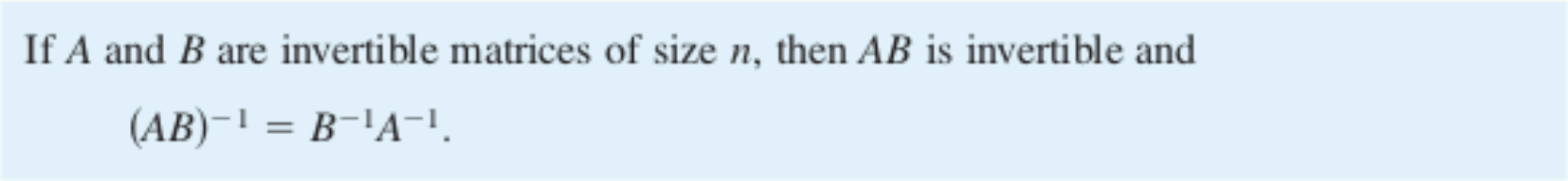
An $n \times n$ matrix $A$ is **invertible** (or **nonsingular**) if there exists an $n \times n$ matrix $B$ such that

$$AB = BA = I_n$$

where $I_n$ is the identity matrix of order $n$. The matrix $B$ is called the (multiplicative) **inverse** of $A$. A matrix that does not have an inverse is called **noninvertible** (or **singular**).



If $A$ is an invertible matrix, then its inverse is unique. The inverse of $A$ is denoted by $A^{-1}$.



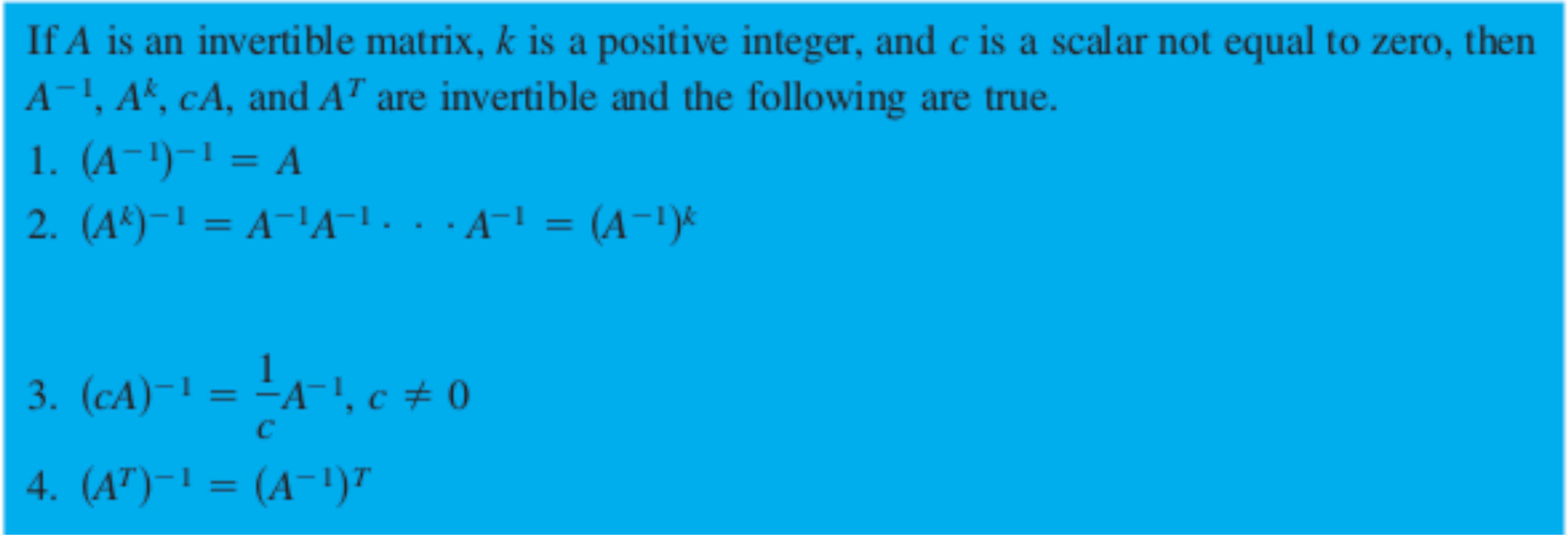
If $A$ and $B$ are invertible matrices of size $n$, then $AB$ is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}.$$

# Dot product vs matrix product: systems, Properties of matrices operations, Inverse and transpose, Elementary matrices and inverses, Applications to graphs:

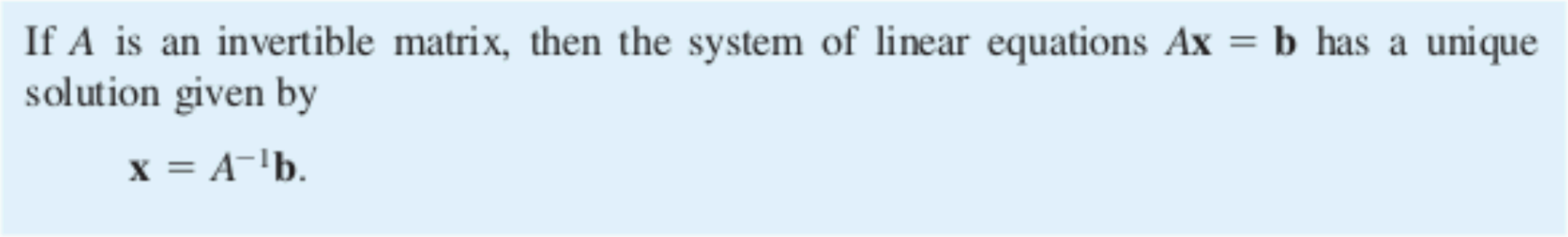
## Matrix Inverse

$$\mathbf{A}_{n\times n} \Rightarrow C = \mathbf{A}^{-1}_{n\times n}\ , \exists \mathbf{A}^{-1} : \mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}_n$$



If $A$ is an invertible matrix, $k$ is a positive integer, and $c$ is a scalar not equal to zero, then $A^{-1}$, $A^k$, $cA$, and $A^T$ are invertible and the following are true.

1. $(A^{-1})^{-1} = A$
2. $(A^k)^{-1} = A^{-1}A^{-1} \cdot \cdot \cdot A^{-1} = (A^{-1})^k$
3. $(cA)^{-1} = \frac{1}{c}A^{-1}, c \neq 0$
4. $(A^T)^{-1} = (A^{-1})^T$



If $A$ is an invertible matrix, then the system of linear equations $A\mathbf{x} = \mathbf{b}$ has a unique solution given by

$$\mathbf{x} = A^{-1}\mathbf{b}.$$

## Finding the inverse by Gauss-Jordan Elimination

1. Given A, form the adjoint matrix $A_{n\times n} \Rightarrow [A \mid I_n]$
2. If possible reduce (through elementary row operations) the adjoint to

$$[A \mid I_n] \Rightarrow [I_n \mid B]$$

then B is the inverse. Otherwise A is singular

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{bmatrix} \quad [A \mid I_3] = \begin{bmatrix} 1 & -1 & 0 & | & 1 & 0 & 0 \\ 1 & 0 & -1 & | & 0 & 1 & 0 \\ -6 & 2 & 3 & | & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - R_1}$$

$$\begin{bmatrix} 1 & -1 & 0 & | & 1 & 0 & 0 \\ 0 & 1 & -1 & | & -1 & 1 & 0 \\ -6 & 2 & 3 & | & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 + 6R_1} \begin{bmatrix} 1 & -1 & 0 & | & 1 & 0 & 0 \\ 0 & 1 & -1 & | & -1 & 1 & 0 \\ 0 & -4 & 3 & | & 6 & 0 & 1 \end{bmatrix}$$

## Dot product vs matrix product: systems, Properties of matrices operations, Inverse and transpose, Elementary matrices and inverses, Applications to graphs:

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & -4 & 3 & 6 & 0 & 1 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 + 4R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 2 & 4 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \leftarrow (-1)R_3} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 + R_3}$$

**Echelon form**

**Starts Jordan: down->up**

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & -3 & -1 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right] \xrightarrow{R_1 \leftarrow R_1 + R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -3 & -1 \\ 0 & 1 & 0 & -3 & -3 & -1 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right]$$

**Inverse exists**

**If the matrix is not invertible, then we will find zeros rows in the left part during the Gauss-Jordan elimination (linear combinations of rows is found) and STOP returning that A is singular**

## Dot product vs matrix product: systems, Properties of matrices operations, Inverse and transpose, Elementary matrices and inverses, Applications to graphs:

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ -2 & 3 & -2 \end{bmatrix} \quad [A \,|\, I_3] = \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 3 & -1 & 2 & 0 & 1 & 0 \\ -2 & 3 & -2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 - 3R_1}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -7 & 2 & -3 & 1 & 0 \\ -2 & 3 & -2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 + 2R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -7 & 2 & -3 & 1 & 0 \\ 0 & 7 & -2 & 2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \leftarrow -\frac{1}{7}R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & -\frac{2}{7} & \frac{3}{7} & -\frac{1}{7} & 0 \\ 0 & 7 & -2 & 2 & 0 & 1 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 - 7R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & -\frac{2}{7} & \frac{3}{7} & -\frac{1}{7} & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{array}\right]$$

**Now, it is impossible to put the identity in the LHS!**

**Elementary matrices**

## Each elementary operation is equivalent to a matrix

**1. Interchange of rows** $R_i \leftrightarrow R_j$ **Is represented by the matrix** $E_{ij}^k$

$$I_{k=3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} E_{23}^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

**Elementary matrices are invertible!**

**2. Multiply a row by a non-zero constant** $R_i \leftarrow cR_i$ **Is represented by** $E_i^k(c)$

$$I_{k=3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \leftarrow cR_2} E_2^3(c) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**2. Add to a row the multiple of another** $R_i \leftarrow R_i + cR_j$ **Is given by** $E_{ij}^k(c)$

$$I_{k=3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 + cR_3} E_{23}^3(c) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$$

# Dot product vs matrix product: systems, Properties of matrices operations, Inverse and transpose, Elementary matrices and inverses, Applications to graphs:

**Elementary matrices**

**To apply an elementary operation just PRE-MULTIPLY the matrix by the corresponding elementary matrix**

$$A = \begin{bmatrix} 2 & -4 & 6 & 2 \\ 1 & -2 & 4 & 3 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \quad E_{12}^{(2)} A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -4 & 6 & 2 \\ 1 & -2 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 4 & 3 \\ 2 & -4 & 6 & 2 \end{bmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_2 - 2R_1} \quad E_{21}^{(2)}(-2) E_{12}^{(2)} A = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 4 & 3 \\ 2 & -4 & 6 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 4 & 3 \\ 0 & -2 & -2 & -4 \end{bmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow (-1/2) R_2} \quad E_{2}^{(2)}(-1/2) E_{21}^{(2)}(-2) E_{12}^{(2)} A = \begin{bmatrix} 1 & 0 \\ 0 & -1/2 \end{bmatrix} \begin{bmatrix} 1 & -2 & 4 & 3 \\ 0 & -2 & -2 & -4 \end{bmatrix} =$$

$$E_{2}^{(2)}(-1/2) E_{21}^{(2)}(-2) E_{12}^{(2)} A = \begin{bmatrix} 1 & -2 & 4 & 3 \\ 0 & 1 & 1 & 2 \end{bmatrix}$$

**Then matrix A is row equivalent to the result of applying all these operations! This is the way that we can implement solving systems and inverses in a matricial way (GPUs)**

# Dot product vs matrix product: systems, Properties of matrices operations, Inverse and transpose, Elementary matrices and inverses, Applications to graphs:

**Applications to graphs**

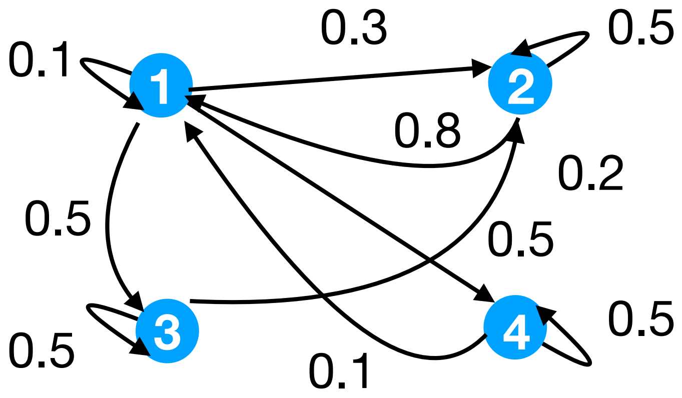
**Matrices can represent graphs but also SYSTEMS WITH A GIVEN BEHAVIOR using PROBABILITIES AS COEFFICENTS**

## Markov Chains

**Suppose that we have a mini-web of n pages. Let $p_{ij}$ be the probability that a web browser staying at page $i$ jumps to page $j$ after a click. A company like Google has collected all these probabilities and stored in a matrix called the transition matrix P. Then given that we start at node $k$, what is the probability of ending at each of the n nodes after a very long number of clicks. Can it be predicted?**

$$P = \begin{array}{c|cccc} & \text{Page1} & \text{Page2} & \text{Page3} & \text{Page4} \\ \hline \text{Page1} & 0.1 & 0.3 & 0.5 & 0.5 \\ \text{Page2} & 0.8 & 0.5 & 0 & 0 \\ \text{Page3} & 0 & 0.2 & 0.5 & 0 \\ \text{Page4} & 0.1 & 0 & 0 & 0.5 \end{array}$$

$$\sum_i p_{ij} = 1 \ \forall j$$



# Dot product vs matrix product: systems, Properties of matrices operations, Inverse and transpose, Elementary matrices and inverses, Applications to graphs:

**Applications to graphs**

## Markov Chains

$$P = \begin{bmatrix} 0.1 & 0.3 & 0.5 & 0.5 \\ 0.8 & 0.5 & 0 & 0 \\ 0.0 & 0.2 & 0.5 & 0 \\ 0.1 & 0 & 0 & 0.5 \end{bmatrix}, \mathbf{x}^0 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$\mathbf{x}^1 = P\mathbf{x}^0$$

$$\mathbf{x}^2 = P\mathbf{x}^1 = P(P\mathbf{x}^0) = P^2\mathbf{x}^0$$

$$\mathbf{x}^3 = P\mathbf{x}^2 = P^3\mathbf{x}^0$$

$$\cdots$$

$$\mathbf{x}^t = P^t\mathbf{x}^0$$

$$\mathbf{x} = P\mathbf{x}$$

**Equilibrium condition**

$$P = \begin{bmatrix} 0.1 & 0.3 & 0.5 & 0.5 \\ 0.8 & 0.5 & 0 & 0 \\ 0.0 & 0.2 & 0.5 & 0 \\ 0.1 & 0 & 0 & 0.5 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \mathbf{x}^1 = \begin{bmatrix} 0.3 \\ 0.5 \\ 0.2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0.1 & 0.3 & 0.5 & 0.5 \\ 0.8 & 0.5 & 0 & 0 \\ 0.0 & 0.2 & 0.5 & 0 \\ 0.1 & 0 & 0 & 0.5 \end{bmatrix} \begin{bmatrix} 0.3 \\ 0.5 \\ 0.2 \\ 0 \end{bmatrix} = \mathbf{x}^2 = \begin{bmatrix} 0.28 \\ 0.49 \\ 0.20 \\ 0.03 \end{bmatrix}$$

$$\mathbf{x} = P\mathbf{x} \Rightarrow I\mathbf{x} = P\mathbf{x}$$

$$\Rightarrow = (I - P)\mathbf{x} = \mathbf{0}$$

Solve an homogeneous system BUT IT HAS A UNIQUE SOLUTION

$$\mathbf{x} = P\mathbf{x}$$

Find eigenvector with eigenvalue 1

# Dot product vs matrix product: systems, Properties of matrices operations, Inverse and transpose, Elementary matrices and inverses, Applications to graphs:

```
def example_Markov():
    from sympy.interactive.printing import init_printing
    init_printing(use_unicode=False, wrap_line=False)
    from sympy.matrices import Matrix
    A= Matrix([[0.1,0.3,0.5,0.5],[0.8,0.5,0,0],[0,0.2,0.5,0],[0.1,0,0,0.5]])
    sol=A.eigenvects()
    return sol
>>>> example_Markov()
```

**TO BE CONTINUED…**