# Linear Algebra for Computer Science

An incremental document:

## From Systems and Matrices to Eigenvalues and Eigenvectors

Francisco Escolano

**Notes based on the text**
**"Elementary Linear Algebra" by Larson and Falvo, 6th edition**

## Formulating & Solving Linear Systems

- Gauss-Jordan
- Homogeneous Systems
- Linear vs non-Linear
- Least squares
- Curve fitting
- Traffic problems
- Dirichlet problems
- Neural Networks
- Echelon form

## Vectors & Matrices

Matrices and Systems

- Properties of matrices
- Product
- Transpose
- Elementary Matrices
- Inverse of a matrix
- Application to Graphs

## Linear Transformations

ONLY THE CONCEPT

- Properties
- Matr
- Kernel
- Isomorphism
- Geometric
- Isometry
- Robo
- Vision
- Graphics

## Vector Spaces & Matrices

Polynomials Lines, planes, hyperplanes

- Spaces and subspaces
- Linear combinations
- Bases and dimension
- Change of basis
- Rank & Nullity
- Dot & Cross products
- Norms and projections
- Stochastic matrices
- PCA

## Eigenvalues and Eigenvectors

Eigenvectors/values and transformations

- Finding eigenpairs
- Eigenspaces
- Quadratic forms and their rotation
- Similarity & Diagonalization
- Graph characterization & PageRank
- Matrix exponentiation
- Systems of Differential equations

Solving an Homogeneous System per eigenvalue

For largest eigenvalue NO NEED OF system solving

$$A\mathbf{x} = \lambda\mathbf{x}$$

Each eigenvalue determines a subspace and the dimensions indicate whether A is diagonalizable

Eigenpairs allow both lossless and lossy changes of basis (PCA)

# Eigenvalues and Eigenvectors

**Eigenvectors/values and transformations**

**Finding eigenpairs** **Eigenspaces**

**Quadratic forms and their rotation**

**Similarity & Diagonalization**

**Graph characterization & PageRank**

**Matrix exponentiation**

**Systems of Differential equations**

Eigenvectors and eigenvalues Define rotation matrices/axes In 2D and 3D

Symmetric Matrices have Real eigenvalues and are diagonalizable

Diagonalization enables new operations In matrices , e.g. expm,() logm(), sin(), some of them are useful in graphs

Spectra define the DNA of graphs & eigenvectors give the steady state of random walks

# 3. Vector spaces and matrices

**Vectors in the plane and in the space (operations), Vector spaces (examples and axioms), Subspaces, Lines and planes, Linear independence and spanned spaces, Base and dimension, Change of basis, Row and column spaces, Rank, Nullspace:**

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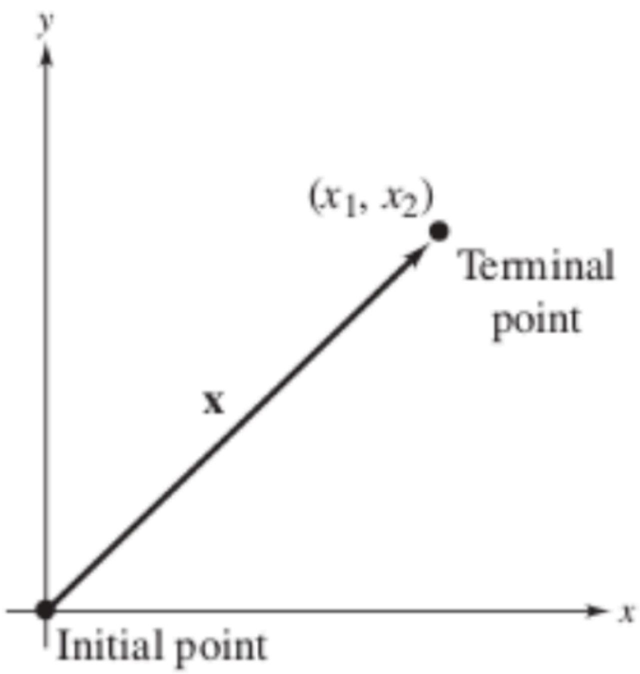
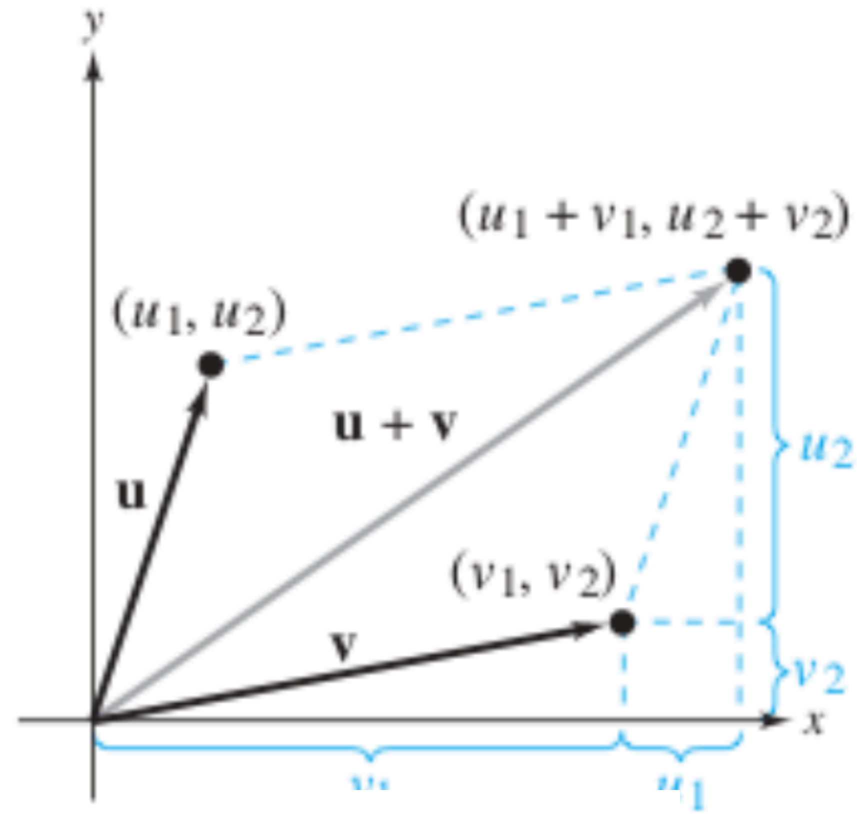
## Vectors in the plane and in the space (operations), Vector spaces (examples and axioms), Subspaces, Lines and planes, linear independence and spanned spaces, Base and dimension, Change of basis, Row and column spaces, Rank, Nullspace:

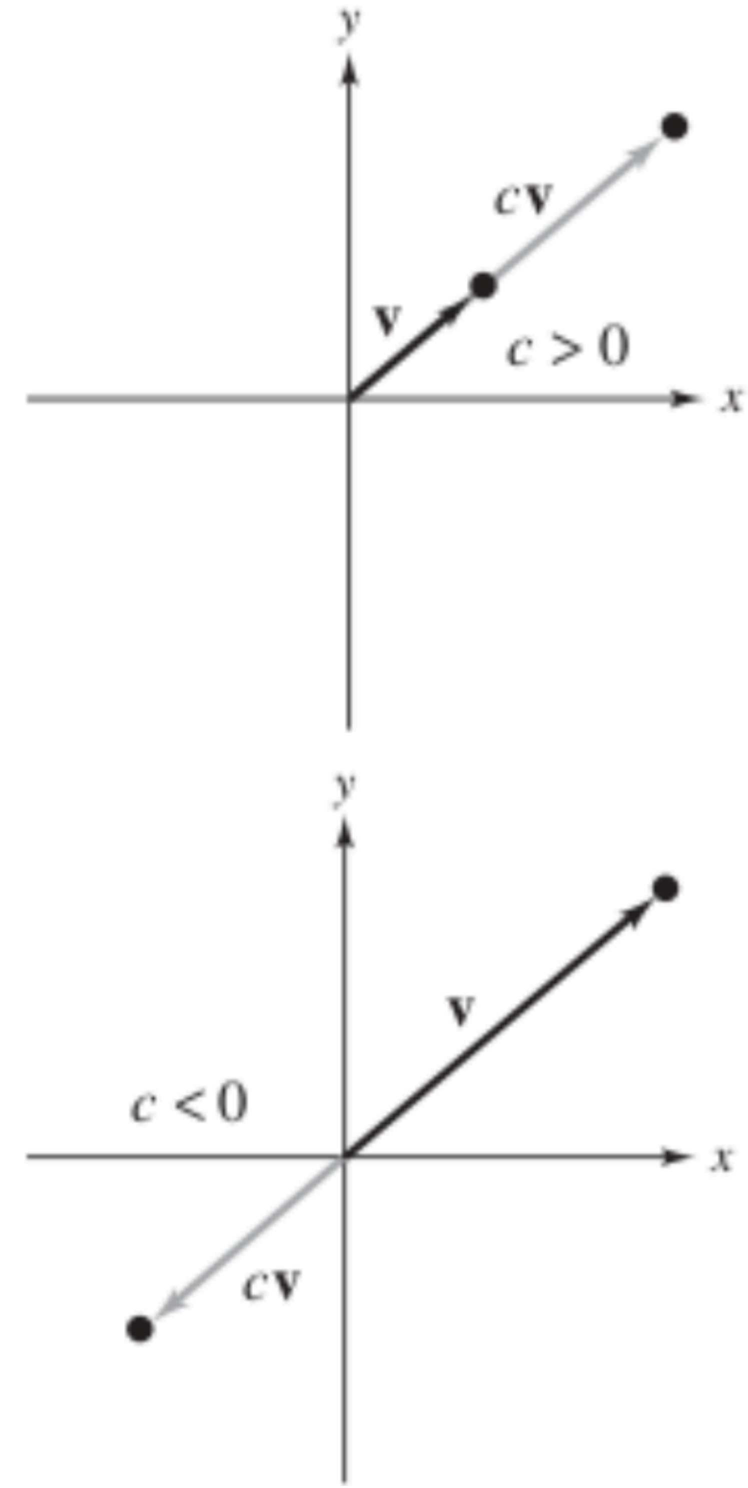
### Vectors in the plane: $\mathbb{R}^2$

$$\mathbf{x} = (x_1, x_2) = [x_1 \ x_2]^T$$




Vector addition



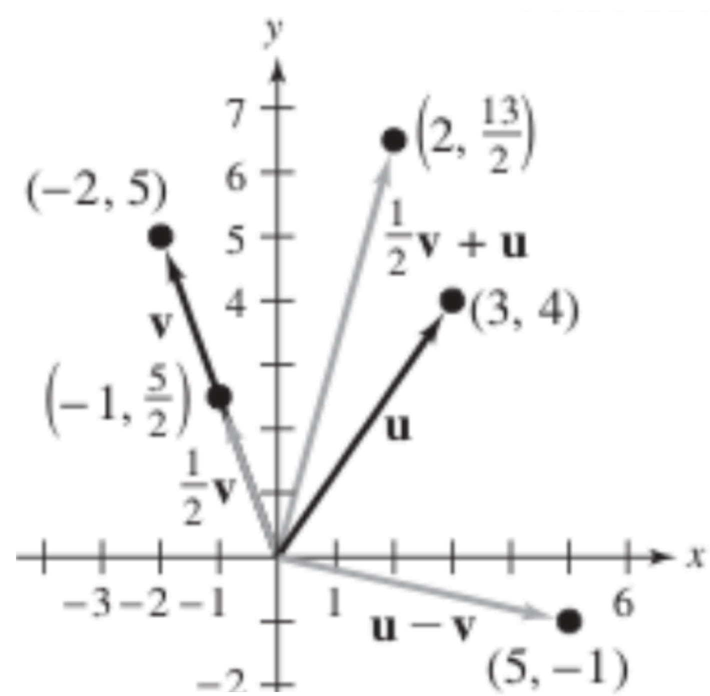

### Basic operations

$$\mathbf{u} = (u_1, u_2), \mathbf{v} = (v_1, v_2)$$

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2)$$

$$c\mathbf{u} = (cu_1, cu_2)$$

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-1)\mathbf{v}$$




Example

Scalar multiplication

## Vectors in the plane and in the space (operations), Vector spaces (examples and axioms), Subspaces, Lines and planes, linear independence and spanned spaces, Base and dimension, Change of basis, Row and column spaces, Rank, Nullspace:

### Vectors in the space: $\mathbb{R}^3$

$$\mathbf{x} = (x_1, x_2, x_3) = [x_1 \; x_2 \; x_3]^T$$

### Basic operations

$$\mathbf{u} = (u_1, u_2, u_3), \mathbf{v} = (v_1, v_2, v_3)$$

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$$

$$c\mathbf{u} = (cu_1, cu_2, cu_3)$$

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-1)\mathbf{v}$$

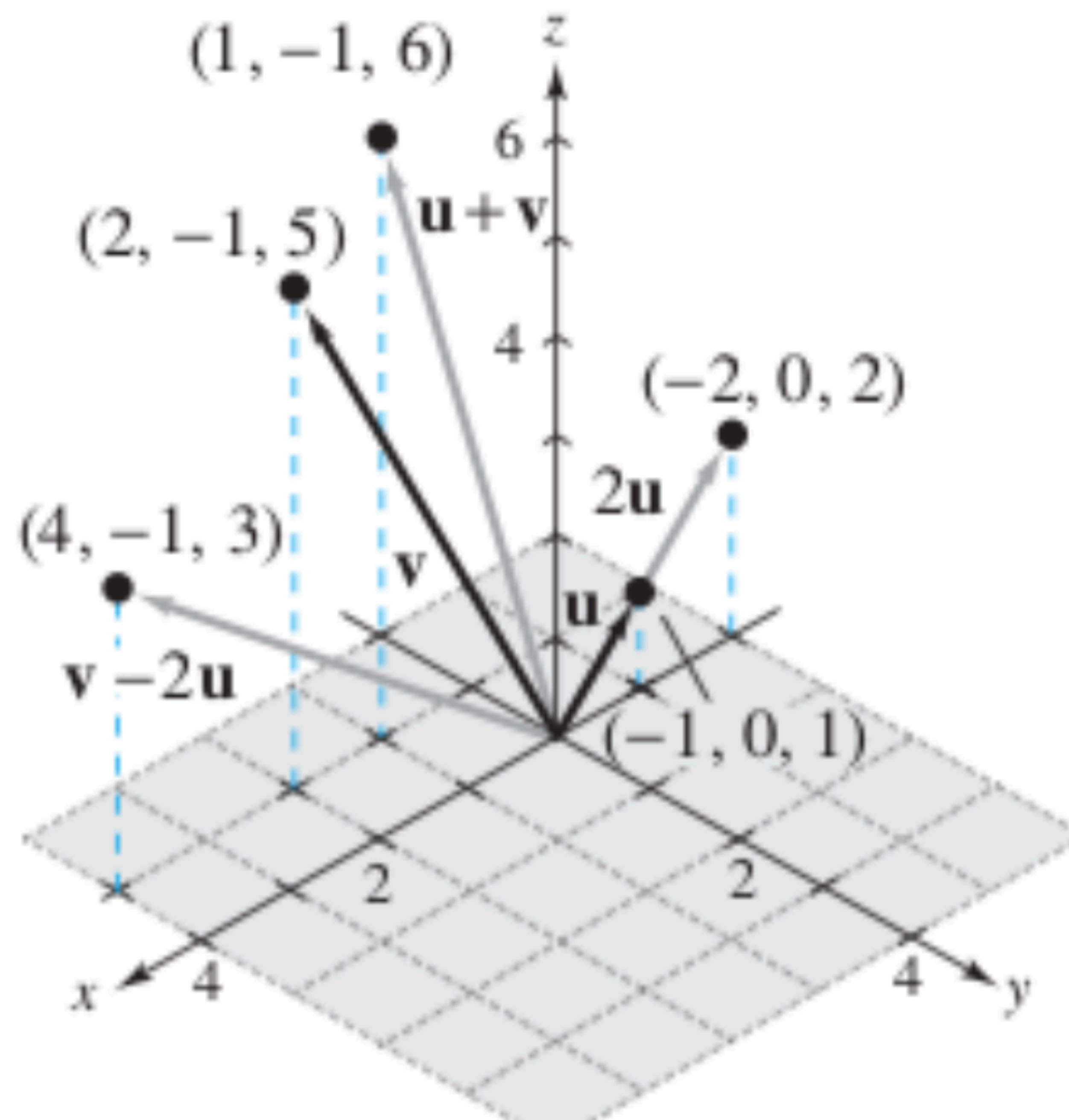
### Main properties
### (a guide to Vector spaces)

$$\mathbf{u} + \mathbf{v} \in \mathbb{R}^3$$

$$c\mathbf{u} \in \mathbb{R}^3$$

$$c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v} \in \mathbb{R}^3$$

$$\mathbf{u} + \mathbf{0} = \mathbf{u} \in \mathbb{R}^3$$




**Example**

## Vectors in the plane and in the space (operations), Vector spaces (examples and axioms), Subspaces, Lines and planes, linear independence and spanned spaces, Base and dimension, Change of basis, Row and column spaces, Rank, Nullspace:

Examples of Vector spaces: $\mathbb{V}$

The concept of vector space is "structural", i.e. it is beyond vector fields in physics with $x \in \mathbb{R}^n$

**Main properties**

$$\mathbf{u} + \mathbf{v} \in \mathbb{V}$$
$$c\mathbf{u} \in \mathbb{V}$$
$$c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v} \in \mathbb{V}$$
$$\mathbf{u} + \mathbf{0} = \mathbf{u} \in \mathbb{V}$$

$\mathscr{M}_{m \times n}$ **Matrices of order $m \times n$**

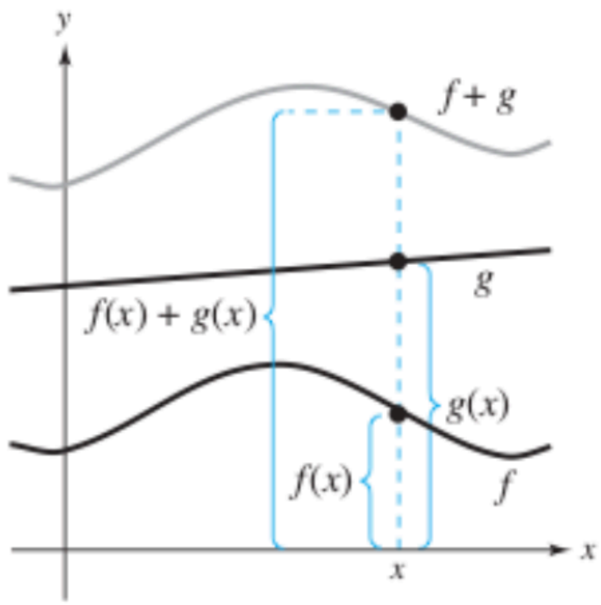
$$A_{m\times n} + B_{m\times n} \in \mathscr{M}_{m\times n}$$

$\mathscr{P}_{k \leq n}$ **Polynomials of degree n or less**

$$p(x) = a_0 + a_1x + a_2x^2$$
$$q(x) = b_0 + b_1x + 0b_2x^2$$
$$p(x) + q(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 \in \mathscr{P}_{k\leq 2}$$

$\mathscr{C}(-\infty, \infty)$ **Continuous functions in $\mathbb{R}$**

$$f(x) + g(x) = (f+g)(x) \in \mathscr{C}, cf(x) \in \mathscr{C}$$
$$sin(x) + cos(x) \in \mathscr{C}$$
$$3sin(x) + e^x \in \mathscr{C}$$



# Vectors in the plane and in the space (operations), Vector spaces (examples and axioms), Subspaces, Lines and planes, linear independence and spanned spaces, Base and dimension, Change of basis, Row and column spaces, Rank, Nullspace:

Axioms for Vector spaces: $\mathbb{V}$

The concept of vector space is "structural", i.e. it is beyond vector fields in physics with $x \in \mathbb{R}^n$

Let $V$ be a set on which two operations (**vector addition** and **scalar multiplication**) are defined. If the listed axioms are satisfied for every $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in $V$ and every scalar (real number) $c$ and $d$, then $V$ is called a **vector space.**

Addition:

| | |
|---|---|
| 1. $\mathbf{u} + \mathbf{v}$ is in $V$. | Closure under addition |
| 2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ | Commutative property |
| 3. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ | Associative property |
| 4. $V$ has a **zero vector 0** such that for every $\mathbf{u}$ in $V$, $\mathbf{u} + \mathbf{0} = \mathbf{u}$. | Additive identity |
| 5. For every $\mathbf{u}$ in $V$, there is a vector in $V$ denoted by $-\mathbf{u}$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$. | Additive inverse |

Scalar Multiplication:

| | |
|---|---|
| 6. $c\mathbf{u}$ is in $V$. | Closure under scalar multiplication |
| 7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ | Distributive property |
| 8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ | Distributive property |
| 9. $c(d\mathbf{u}) = (cd)\mathbf{u}$ | Associative property |
| 10. $1(\mathbf{u}) = \mathbf{u}$ | Scalar identity |

## Vectors in the plane and in the space (operations), Vector spaces (examples and axioms), Subspaces, Lines and planes, linear independence and spanned spaces, Base and dimension, Change of basis, Row and column spaces, Rank, Nullspace:

Examples of NON Vector spaces

**Usually verify the CLOSURE PROPERTY**

$$\mathbf{u} + \mathbf{v} \in \mathbb{V}$$
$$c\mathbf{u} \in \mathbb{V}$$
$$c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v} \in \mathbb{V}$$
$$\mathbf{u} + \mathbf{0} = \mathbf{u} \in \mathbb{V}$$

$\mathscr{P}_n$ **Polynomials of exact degree n**

$$p(x) = a_0 + a_1 x + a_2 x^2$$
$$q(x) = b_0 + b_1 x - a_2 x^2$$
$$p(x) + q(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 - a_2)x^2 \in \mathscr{P}_2$$

**Set of matrices with a special form:**

$$\mathbb{V} = \left\{ \begin{bmatrix} a & b \\ c & 1 \end{bmatrix} \right\}$$

$$\begin{bmatrix} a_1 & b_1 \\ c_1 & 1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & 1 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & 2 \end{bmatrix} \notin \mathbb{V}$$

$$\begin{bmatrix} a_1 & b_1 \\ c_1 & 0 \end{bmatrix} \notin \mathbb{V}$$

Vectors in the plane and in the space (operations), Vector spaces (examples and axioms), Subspaces, Lines and planes, linear independence and spanned spaces, Base and dimension, Change of basis, Row and column spaces, Rank, Nullspace:

Vector Subspaces $\mathbb{W}$

INHERITANCE OF THE PROPERTIES OF VECTOR SPACES. For instance 0 must belong to the subspace!

**Axioms**

$$\mathbf{u} + \mathbf{v} \in \mathbb{V}$$

$$c\mathbf{u} \in \mathbb{V}$$

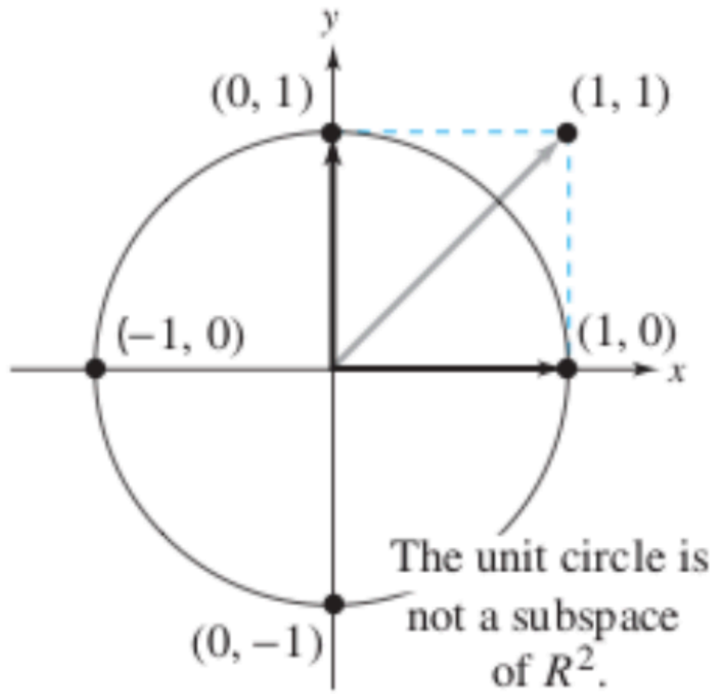
## LACK OF ZERO ELEMENT:

$$\mathbb{W} = \{(x_1, x_2, x_3) : x_1 + x_2 + x_3 = 1\}$$

## LACK OF CLOSURE:

$$\mathbb{W} = \{(x_1, x_2) : x_1^2 + x_2^2 = 1\}$$

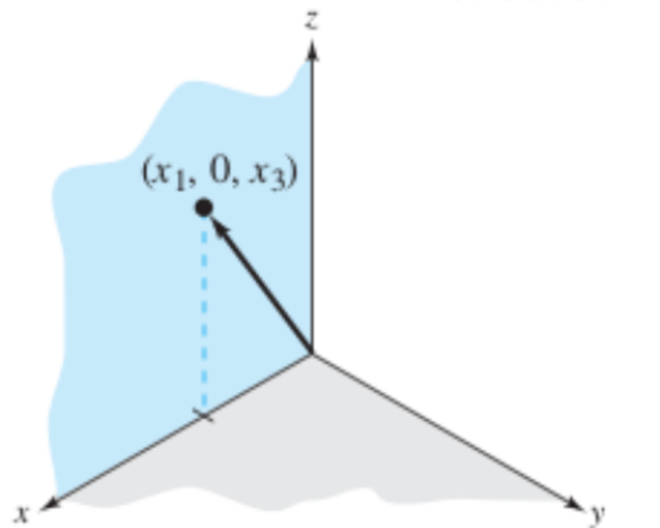
$$(0,0) \in \mathbb{W} \quad (1,1) \notin \mathbb{W} : 1^2 + 1^2$$




## Some Projections

$$\mathbb{W} = \{(x_1, 0, x_3)\}$$

SOLUTION



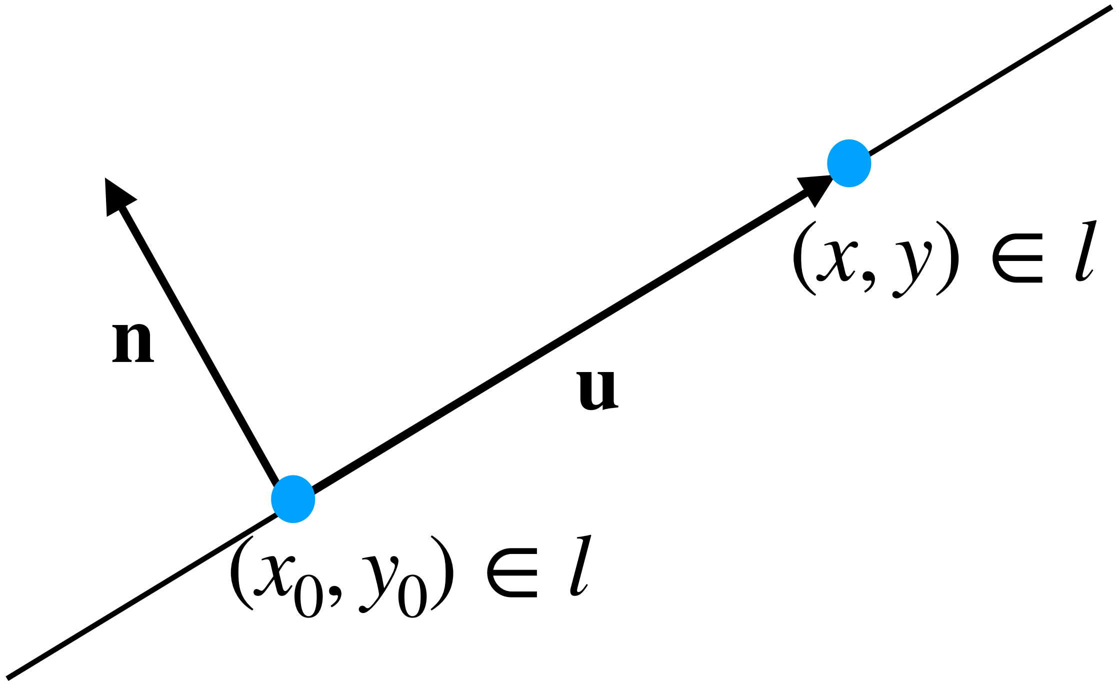
**Vectors in the plane and in the space (operations), Vector spaces (examples and axioms), Subspaces, Lines and planes,** linear independence and spanned spaces, Base and dimension, Change of basis, Row and column spaces, Rank, Nullspace:

**Lines in the plane. Are subspaces of $\mathbb{R}^2$?**

$$l : ax_0 + by_0 + c = 0$$

$$c = -(ax_0 + by_0)$$

$$(x_0, y_0) \in l$$




$(x, y) \in V_l$ **if** $ax + by + c = 0$

**ONLY IF** $(0,0) \in l \Leftrightarrow c = 0$

$$ax + by - (ax_0 + bx_0) = 0$$

$$a(x - x_0) + b(y - y_0) = 0$$

$$< (a, b), (x - x_0, y - y_0) > = 0$$

$$< \mathbf{n}, \mathbf{u} > = 0$$

$$\mathbf{n} = (a, b), \mathbf{u} = (x - x_0, y - y_0)$$

**Check whether $V_l \subset \mathbb{R}^2$**

$\mathbf{u}, \mathbf{v} \in V_l : \mathbf{u} + \mathbf{v} \in V_l$**?**

$$< \mathbf{n}, \mathbf{u} > = 0, < \mathbf{n}, \mathbf{v} > = 0$$

$$< \mathbf{n}, (\mathbf{u} + \mathbf{v}) > = < \mathbf{n}, \mathbf{u} > + < \mathbf{n}, \mathbf{v} > = 0$$

$\mathbf{u} \in V_l : c\mathbf{u} \in V_l$**?**

$$< \mathbf{n}, c\mathbf{u}) > = c < \mathbf{n}, \mathbf{u} > = 0$$

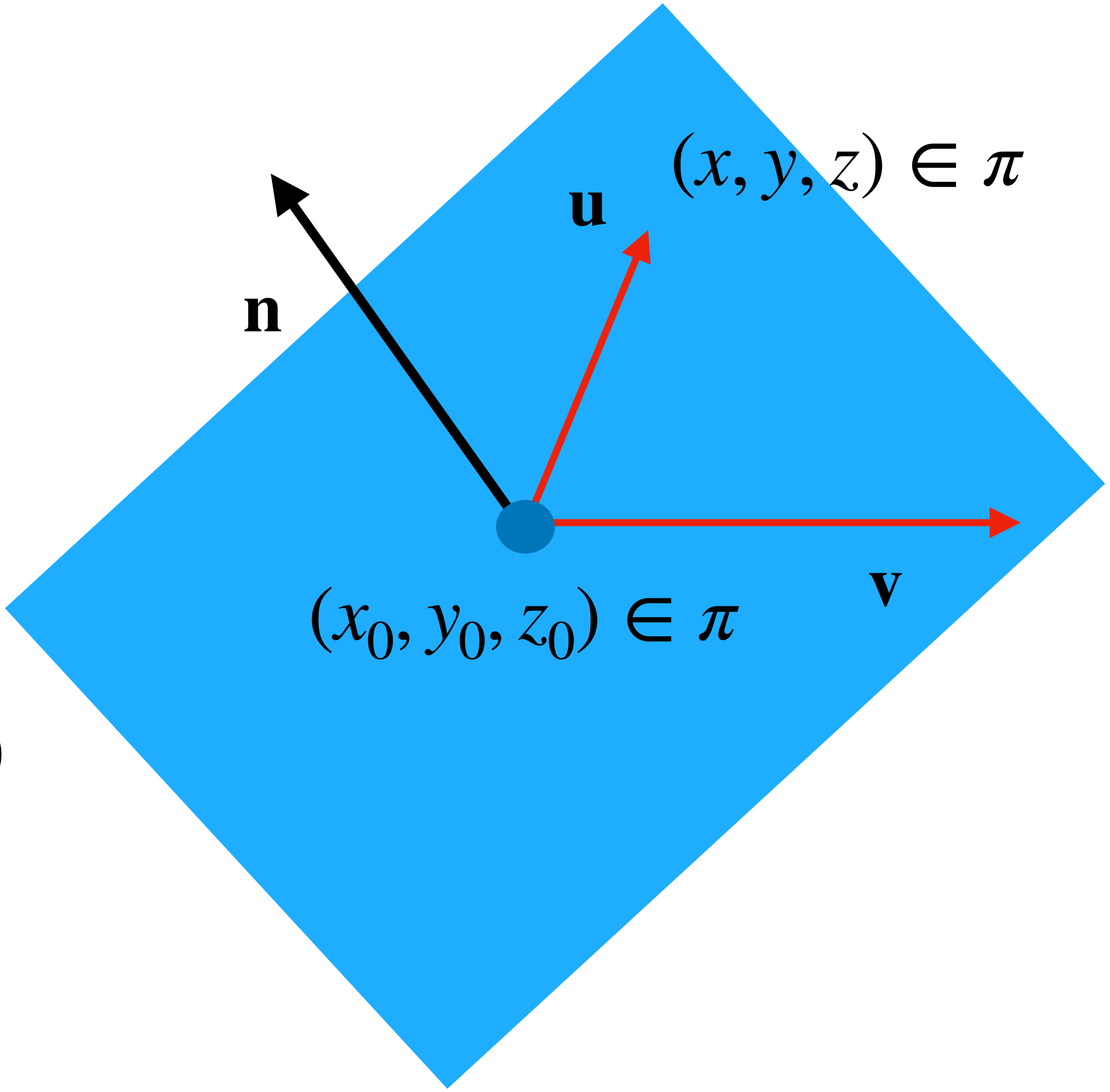
Vectors in the plane and in the space (operations), Vector spaces (examples and axioms), Subspaces, Lines and planes, linear independence and spanned spaces, Base and dimension, Change of basis, Row and column spaces, Rank, Nullspace:

**Planes in the space**
**Are subspaces of $\mathbb{R}^3$?**

$$\pi : ax_0 + by_0 + cz_0 + d = 0$$

$$d = -(ax_0 + by_0 + cz_0)$$

$$(x_0, y_0, z_0) \in \pi$$




$(x, y, z) \in V_\pi$ **if** $ax + by + cz + d = 0$

$$ax + by + cz - (ax_0 + bx_0 + cz_0) = 0$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$< (a, b, c), (x - x_0, y - y_0, z - z_0) > = 0$$

$$< \mathbf{n}, \mathbf{u} > = 0$$

$$\mathbf{n} = (a, b, c), \mathbf{u} = (x - x_0, y - y_0, z - z_0)$$

**ONLY IF** $(0,0,0) \in \pi \Leftrightarrow c = 0$

**Check whether**

$\mathbf{u}, \mathbf{v} \in V_\pi : \mathbf{u} + \mathbf{v} \in V_\pi$**?**

$\mathbf{u} \in V_\pi : c\mathbf{u} \in V_\pi$**?**

Vectors in the plane and in the space (operations), Vector spaces (examples and axioms), Subspaces, Lines and planes, linear independence and spanned spaces, Base and dimension, Change of basis, Row and column spaces, Rank, Nullspace:

Linear combination

## Definition and finding

1. Given $\mathbf{v} \in \mathbb{V}$, we say that it is a linear combination of vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k \in \mathbb{V}$, if there exist scalars $c_1, c_2, \ldots, c_k$ so that

$$\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \ldots + c_k\mathbf{u}_k$$

2. For finding, just set a linear system where the unknowns are the scalars that and check that it is consistent. If so, we have a linear combination (or many).

$\mathbf{v} = (1,3,1)$ vs $S = \{(1,2,3), (0,1,2), (-1,0,1)\}$?

**Solution:**

$$\begin{bmatrix}1\\3\\1\end{bmatrix} = c_1\begin{bmatrix}1\\2\\3\end{bmatrix} + c_2\begin{bmatrix}0\\1\\2\end{bmatrix} + c_3\begin{bmatrix}-1\\0\\1\end{bmatrix} \Rightarrow \begin{bmatrix}1 & 0 & -1\\2 & 1 & 0\\3 & 2 & 1\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}1\\3\\1\end{bmatrix}$$

$$c_1 = 1 + t$$
$$c_2 = -1 - 2t$$
$$c_3 = t$$

$\mathbf{v} = \begin{bmatrix}0 & 8\\2 & 1\end{bmatrix}$ vs $S = \left\{\begin{bmatrix}0 & 2\\1 & 0\end{bmatrix}, \begin{bmatrix}-1 & 3\\1 & 2\end{bmatrix}, \begin{bmatrix}-2 & 0\\1 & 3\end{bmatrix}\right\}$?

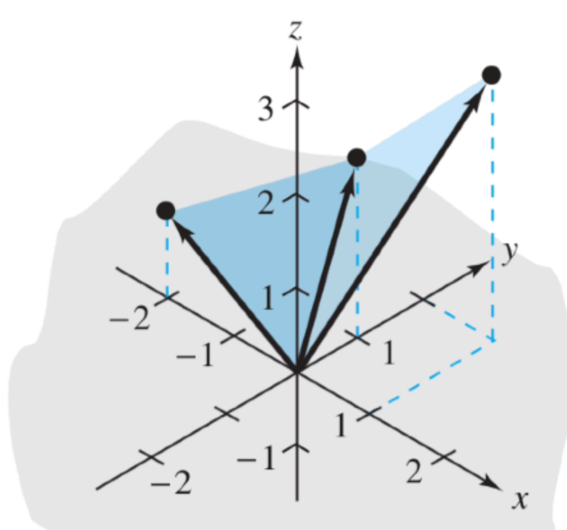
First VECTORTIZE?

## Spanning sets

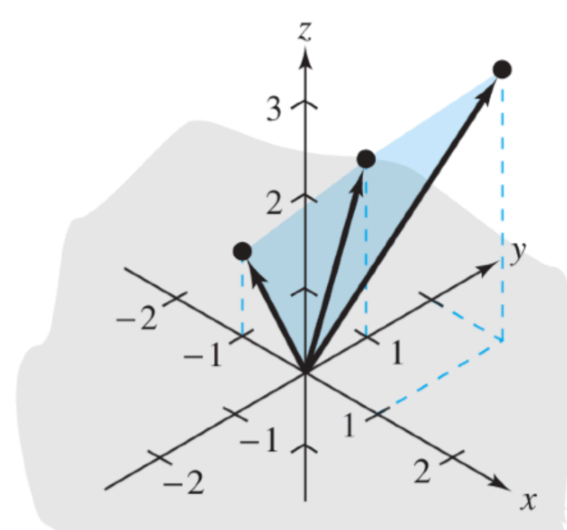
### Definition

1. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} \subseteq \mathbb{V}$, this set is called a spanning set if every vector in the subspace $\mathbb{V}$ can be written as a linear combination of the elements of $S$. In this case, it is said that the set spans $\mathbb{V}$, i.e. $\mathbb{V}$=span($S$).

2. The span of a set $S$, span(S)=$\{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k\}$ is the set of all the linear combinations of $S$




$S_1 = \{(1, 2, 3), (0, 1, 2), (-2, 0, 1)\}$
The vectors in $S_1$ do not lie in a common plane.




$S_2 = \{(1, 2, 3), (0, 1, 2), (-1, 0, 1)\}$
The vectors in $S_2$ lie in a common plane.

Linear independence

## Definition and finding

1. A set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} \subseteq \mathbb{V}$, is linearly independent if the vector equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k = \mathbf{0}$ has a unique solution (the trivial one). Otherwise it is linearly dependent.

2. To determine linear independence, just solve the homogeneous system.

$$S = \{(1,2,3), (0,1,2), (-2,0,1)\}$$

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$S = \{1 + x - 2x^2, 2 + 5x - x^2, x + x^2\}$ **?**

## Basis and dimension

### Definitions and finding

1. A set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} \subseteq \mathbb{V}$, is called a basis of $\mathbb{V}$ if

   a) It spans the whole space $\mathbb{V}$ and

   b) $S$ is linearly independent

2. If a vector space $\mathbb{V}$ has a basis of $n$ vectors, then its dimension is dim($\mathbb{V}$)=$n$
   If the vector space consists only of the zero vector, then the dimension is 0.

$S = \{(d, c - d, c) : c, d \in \mathbb{R}\}$**?**

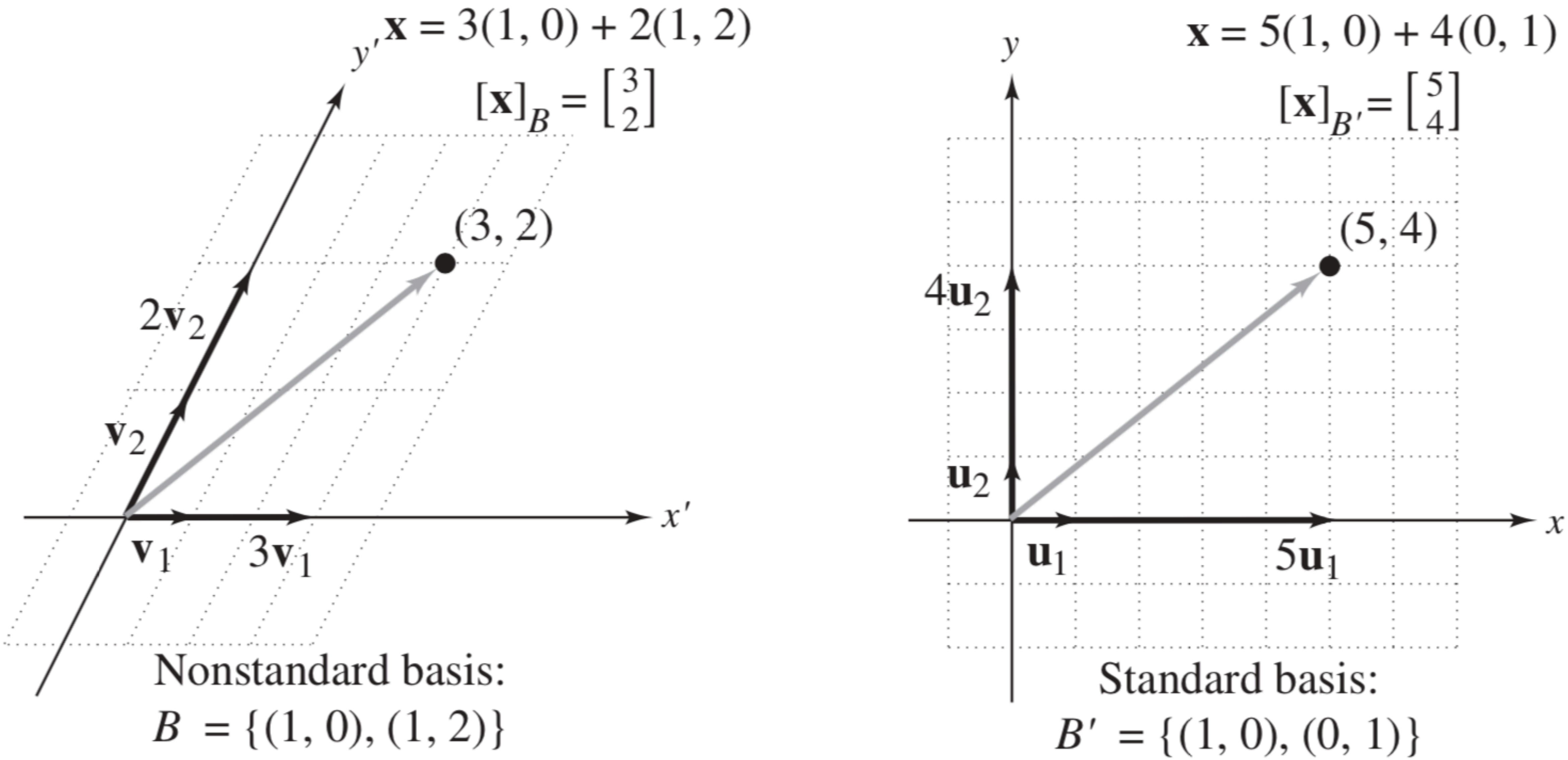
$$(d, c - d, c) = (0,c, c) + (d, -c,0) = c(0,1,1) + d(1, -1,0) \Rightarrow \textbf{dim}(S) = 2$$

$S = \{(2b, b,0) : b \in \mathbb{R}\}$**?**

$$(2b, b,0) = b(2,1,0) \Rightarrow \textbf{dim}(S) = 1$$

## Vectors in the plane and in the space (operations), Vector spaces (examples and axioms), Subspaces, linear independence and spanned spaces, Base and dimension, Change of basis, Row and column spaces, Rank, Nullspace:

**Change of basis**




IN THIS CASE we assume that the origin of coordinates does not change

## Vectors in the plane and in the space (operations), Vector spaces (examples and axioms), Subspaces, linear independence and spanned spaces, Base and dimension, Change of basis, Row and column spaces, Rank, Nullspace:

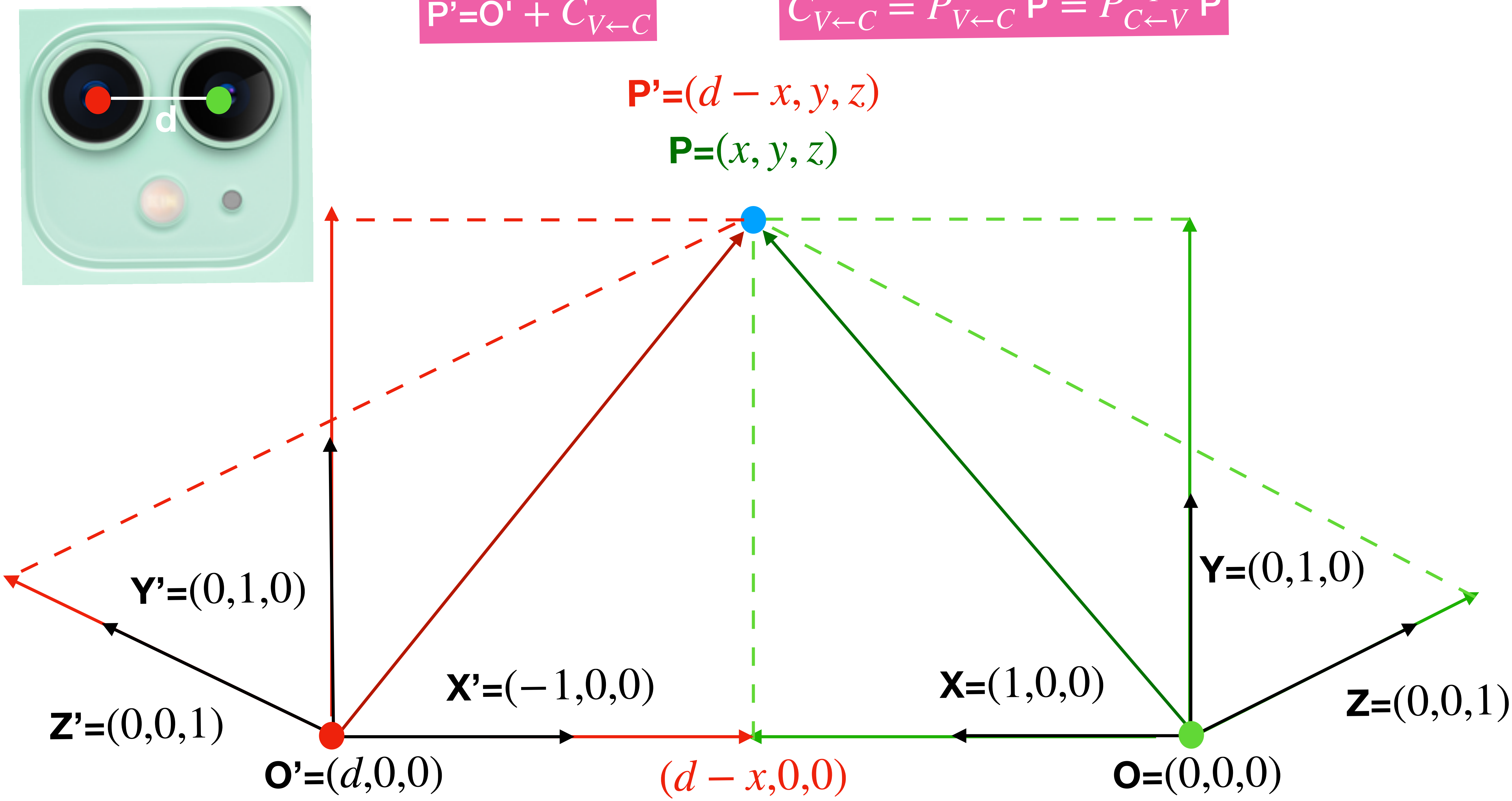
Change of basis

Coordinates in the left camera come from the origin of the left camera + The change of coordinates between the two bases

### Stereo Camera

P'=O' + $C_{V \leftarrow C}$

$C_{V \leftarrow C} = P_{V \leftarrow C}$ P = $P^{-1}_{C \leftarrow V}$ P



## Vectors in the plane and in the space (operations), Vector spaces (examples and axioms), Subspaces, linear independence and spanned spaces, Base and dimension, Change of basis, Row and column spaces, Rank, Nullspace:

P'=O' + $C_{V \leftarrow C}$

$C_{V \leftarrow C} = P_{V \leftarrow C}$ P = $P^{-1}_{C \leftarrow V}$ P

**Coordinates of the canonical base C In the new base V**

**Change of coordinates from C to the new base V**

**Matrix Inversion to Change of coordinates from V to the canonical base C**

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} d \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} d \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

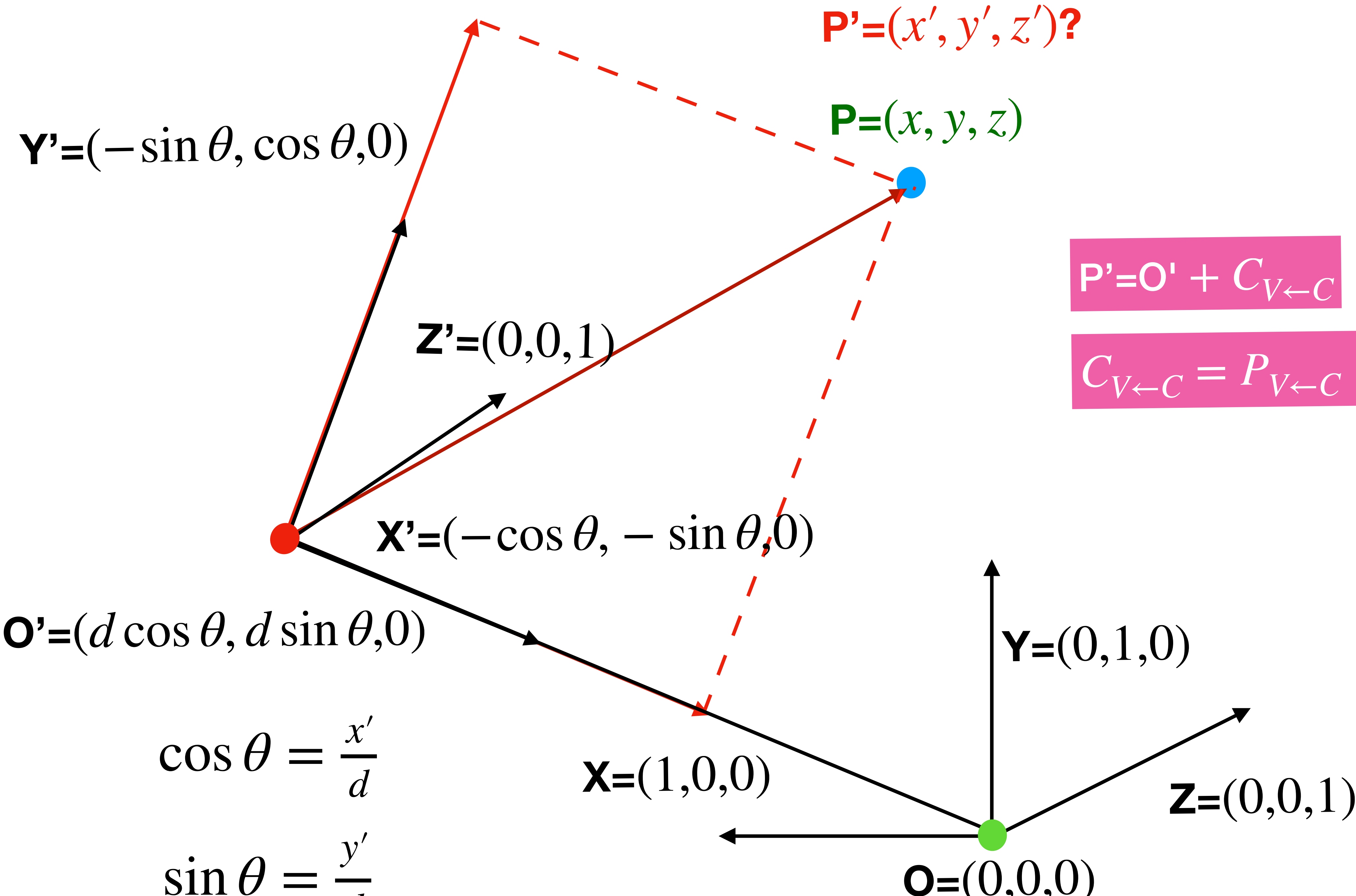
$$= \begin{bmatrix} d \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d - x \\ y \\ z \end{bmatrix}$$

## Vectors in the plane and in the space (operations), Vector spaces (examples and axioms), Subspaces, linear independence and spanned spaces, Base and dimension, Change of basis, Row and column spaces, Rank, Nullspace:

Change of basis

Coordinates once we rotate the left camera around the Z axis and holding X'Y' coplanar with XY

### Stereo Camera




P'=O' + $C_{V\leftarrow C}$

$C_{V\leftarrow C} = P_{V\leftarrow C}$ P $= P^{-1}_{C\leftarrow V}$ P

$$\cos\theta = \frac{x'}{d}$$

$$\sin\theta = \frac{y'}{d}$$

## Vectors in the plane and in the space (operations), Vector spaces (examples and axioms), Subspaces, linear independence and spanned spaces, Base and dimension, Change of basis, Row and column spaces, Rank, Nullspace:

P'=O' + $C_{V \leftarrow C}$

$C_{V \leftarrow C} = P_{V \leftarrow C}$ P = $P^{-1}_{C \leftarrow V}$ P

**Coordinates of the canonical base C In the new base V**

**Change of coordinates from C to the new base V**

**Matrix Inversion to Change of coordinates from V to the canonical base C**

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} d\cos\theta \\ d\sin\theta \\ 0 \end{bmatrix} + \begin{bmatrix} -\cos\theta & -\sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} d\cos\theta \\ d\sin\theta \\ 0 \end{bmatrix} + \begin{bmatrix} -\cos\theta & -\sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} d\cos\theta \\ d\sin\theta \\ 0 \end{bmatrix} + \begin{bmatrix} -x\cos\theta - y\sin\theta \\ -x\sin\theta + y\cos\theta \\ z \end{bmatrix}$$

Change of basis

## Definition and finding

1. Given two bases $\mathbf{V} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, $\mathbf{U} = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ of $\mathbb{V}$, then if

$$\mathbf{v}_1 = c_{11}\mathbf{u}_1 + c_{21}\mathbf{u}_2 + \ldots + c_{n1}\mathbf{u}_n$$
$$\mathbf{v}_2 = c_{12}\mathbf{u}_1 + c_{22}\mathbf{u}_2 + \ldots + c_{n2}\mathbf{u}_n$$
$$\vdots$$
$$\mathbf{v}_n = c_{1n}\mathbf{u}_1 + c_{2n}\mathbf{u}_2 + \ldots + c_{nn}\mathbf{u}_n$$

Then the matrix of coefficients $P_{U\leftarrow V} = [c_{ij}]$ is the transition matrix from $\mathbf{V}$ to $\mathbf{U}$

2. Then the transition matrix $P_{V\leftarrow U}$ from $\mathbf{V}$ to $\mathbf{U}$ does exist and it is equal to $P^{-1}_{U\leftarrow V}$

3. If none of the bases is the canonical (standard) base $\mathscr{C}$, then $P_{U\leftarrow V}$ is

$$P_{U\leftarrow V} = P_{U\leftarrow \mathscr{C}}P_{\mathscr{C}\leftarrow V} = P^{-1}_{\mathscr{C}\leftarrow U}P_{\mathscr{C}\leftarrow V}$$

## Example#1: Coordinates of $\mathbf{x} = (1,2,3)$ in the basis

$$\mathbf{U} = \{(1,0,-2),(3,-1,2),(-2,-2,1)\}$$

$$C_U(\mathbf{x}) = P_{U\leftarrow\mathscr{C}}C_{\mathscr{C}}(\mathbf{x}) = P^{-1}_{\mathscr{C}\leftarrow\mathscr{U}}C_{\mathscr{C}}(\mathbf{x})$$

$$C_U(\mathbf{x}) = \begin{bmatrix} 1 & 3 & -2 \\ 0 & -1 & -2 \\ -2 & 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -\frac{28}{11} \\ -\frac{1}{11} \\ -\frac{21}{11} \end{bmatrix}$$

## Example#2: Change V to U

$$\mathbf{U} = \{(1,1,0),(1,0,1),(0,1,1)\} \quad \mathbf{V} = \{(1,1,1),(-1,1,1),(0,2,1)\}$$

$$P_{U\leftarrow V} = P_{U\leftarrow\mathscr{C}}P_{\mathscr{C}\leftarrow V} = P^{-1}_{\mathscr{C}\leftarrow U}P_{\mathscr{C}\leftarrow V}$$

$$P_{U\leftarrow U} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{3}{2} & \frac{1}{2} \\ -1 & \frac{1}{2} & \frac{1}{2} \\ 1 & -1 & 0 \end{bmatrix}$$

**Example#3: in $\mathscr{P}_2$ the canonical base is $\mathscr{C} = \{1, x, x^2\}$ then, given the basis $B = \{4x - 1, 2x^2 - x, 3x^2 + 3\}$, find the coordinates of the poly $p(x) = a_0 + a_1 x + a_2 x^2$ in $B$**

**Is $B$ a basis?** $B = \{4x - 1, 2x^2 - x, 3x^2 + 3\}$

$B = \{(-1,4,0), (0,-1,2), (3,0,3)\}$

Solve in Sympy

$$\begin{bmatrix} -1 & 0 & 3 \\ 4 & -1 & 0 \\ 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

**Must have a unique solution (the trivial one): YES**

$C_B(5x^2 - 3x + 4)$**?**

$$C_B(p(x)) = P_{B \leftarrow \mathscr{C}} C_{\mathscr{C}}(p(x)) = P^{-1}_{\mathscr{C} \leftarrow B} C_{\mathscr{C}}(p(x))$$

$$C_B(p(x)) = \begin{bmatrix} -1 & 0 & 3 \\ 4 & -1 & 0 \\ 0 & 2 & 3 \end{bmatrix}^{-1} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{27}(-3a_0 + 6a_1 + 3a_2) \\ \frac{1}{27}(-12a_0 - 3a_1 + 12a_2) \\ \frac{1}{27}(8a_0 - 2a_1 + a_2) \end{bmatrix}$$

Row and column spaces

A matrix spans both a ROW space and a COLUMN space

## Definition and finding a base

1. **Given a $m \times n$ matrix $A$, the space spanned by its rows is found by finding a matrix $B$ with is row equivalent to $A$ and taking its non-zero rows as a basis**

2. **Given a $m \times n$ matrix $A$, the space spanned by its cols is found by finding a matrix $B$ with is row equivalent to $A^T$ and taking its non-zero rows as a basis**

$$A = \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 3 \\ -3 & 0 & 6 & 1 \\ 3 & 4 & -2 & 1 \\ 2 & 0 & -4 & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & -3 & 3 & 2 \\ 0 & 1 & 9 & -5 & -6 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Row space     Column space

# Vectors in the plane and in the space (operations), Vector spaces (examples and axioms), Subspaces, linear independence and spanned spaces, Base and dimension, Change of basis, Row and column spaces, Rank, Nullspace:

Rank of a matrix

Rank is the dimension of the row (column) space

## Definition and finding

1. Given a $m \times n$ matrix $A$, the both the row and col space is have the same dimension and it is call the rank of $A$: rank($A$).

2. Some properties: rank($A$) $\leq \min(m, n)$

$$A = \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 3 \\ -3 & 0 & 6 & 1 \\ 3 & 4 & -2 & 1 \\ 2 & 0 & -4 & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & -3 & 3 & 2 \\ 0 & 1 & 9 & -5 & -6 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Row space | Column space

rank($A$)=dim(Row($A$))=3=dim(Col($A$))=3 $\leq \min(5,4) = 4$

Rank of a matrix

Rank is the dimension of the row (column) space

## Definition and finding

1. **Given a $m \times n$ matrix $A$, the both the row and col space is have the same dimension and it is call the rank of $A$: rank($A$).**

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$$A = \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 3 \\ -3 & 0 & 6 & 1 \\ 3 & 4 & -2 & 1 \\ 2 & 0 & -4 & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & -3 & 3 & 2 \\ 0 & 1 & 9 & -5 & -6 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Row space Column space

**rank($A$)=dim(Row($A$))=3=dim(Col($A$))=3 $\leq \min(5,4) = 4$**

## Vectors in the plane and in the space (operations), Vector spaces (examples and axioms), Subspaces, linear independence and spanned spaces, Base and dimension, Change of basis, Row and column spaces, Rank, Nullspace:

Nullspace

Set of solutions of the homogeneous system

### Definitions and finding

1. Given a $m \times n$ matrix $A$, the null space Null($A$) is the set of solutions of $A\mathbf{x} = \mathbf{0}$
2. The dimension of the null space is the nullity of $A$: nullity($A$)=dim(Null($A$))
3. Fundamental property: rank($A$)+nullity($A$)=$n$ (i.e. dim(Null($A$)=n-rank($A$))
4. If $\mathbf{x}$ is a solution of $A\mathbf{x} = \mathbf{b}$, then it can be written as $\mathbf{x} = \mathbf{x}_P + \mathbf{x}_H$ where the first term is a particular solution of $A\mathbf{x} = \mathbf{b}$ and the second a solution of $A\mathbf{x} = \mathbf{0}$

$$\begin{bmatrix} 1 & 0 & -2 & 1 & | & 5 \\ 3 & 1 & -5 & 0 & | & 8 \\ 1 & 2 & 0 & -5 & | & 9 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -2 & 1 & 5 \\ 0 & 1 & 1 & -3 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow s \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 3 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 5 \\ -7 \\ 0 \\ 0 \end{bmatrix}$$

# Vectors in the plane and in the space (operations), Vector spaces (examples and axioms), Subspaces, linear independence and spanned spaces, Base and dimension, Change of basis, Row and column spaces, Rank, Nullspace:

**TO BE CONTINUED…**