Linear Algebra for Computer Science

An incremental document:

From Systems and Matrices to Eigenvalues and Eigenvectors

Francisco Escolano

Notes based on the text
“Elementary Linear Algebra” by Larson and Falvo, 6th edition

Formulating & Solving Linear Systems

Linear vs non-Linear

Gauss-Jordan

Homogeneous Systems

Least squares
Curve fitting
Traffic problems
Dirichlet problems
Neural Networks

Echelon form

Vectors & Matrices

Matrices and Systems

Properties of matrices
Product
Transpose

Elementary Matrices

Inverse of a matrix

Application to Graphs

Linear Transformations

Properties
Matricial rep
Kernel and Range
Isomorphism
Geometric
Isometry

Robotics
Vision
Graphics

Vector Spaces & Matrices

Polynomials Lines, planes, hyperplanes

Spaces and subspaces
Linear combinations
Bases and dimension
Change of basis

Rank & Nullity

Dot & Cross products
Norms and projections

Stochastic matrices
PCA

Eigenvalues and Eigenvectors

Eigenvectors/values and transformations

Finding eigenpairs

Eigenspaces

Quadratic forms and their rotation

Similarity & Diagonalization

Graph characterization & PageRank

Matrix exponentiation

Systems of Differential equations

Solving an Homogeneous System per eigenvalue

For largest eigenvalue NO NEED OF system solving

$$A\mathbf{x} = \lambda \mathbf{x}$$

Each eigenvalue determines a subspace and the dimensions indicate whether A is diagonalizable

Eigenpairs allow both lossless and lossy changes of basis (PCA)

Eigenvalues and Eigenvectors

Eigenvectors/values and transformations

Finding eigenpairs

Eigenspaces

Quadratic forms and their rotation

Similarity & Diagonalization

Graph characterization & PageRank

Matrix exponentiation

Systems of Differential equations

Eigenvectors and eigenvalues Define rotation matrices/axes In 2D and 3D

Symmetric Matrices have Real eigenvalues and are diagonalizable

Diagonalization enables new operations In matrices , e.g. expm,() logm(), sin(), some of them are useful in graphs

Spectra define the DNA of graphs & eigenvectors give the steady state of random walks

5. Eigenvalues and Eigenvectors

Intuition, Finding eigenpairs, Eigenspaces, Matrix similarity and diagonalization, Spectral theorem, Applications:

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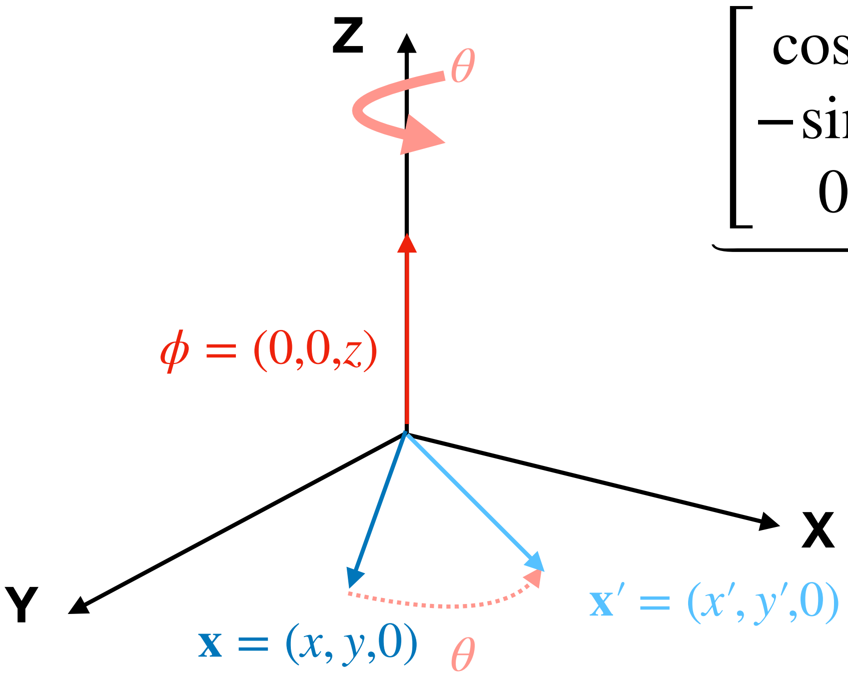
Intuition, Finding eigenpairs, Eigenspaces, Matrix similarity and diagonalization, Spectral theorem, Applications:

Matrices are OPERATORS

$$A\mathbf{x} = ? \qquad A_{n\times n}\mathbf{x}_{n\times 1} = ?$$

MATRICES A OPERATE ON VECTORS TO TRANSFORM THEM

A is a Rotation matrix $\quad A\mathbf{x} = \mathbf{x}'$



$$\underbrace{\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} x\cos\theta + y\sin\theta \\ -x\sin\theta + y\cos\theta \\ z \end{bmatrix}}_{\mathbf{x}'}$$

Points in the XY plane have z=z'=0

Points on the Z axis do not rotate

$$A\phi = \lambda\phi,\ \lambda = 1,\ \phi = [0\ 0\ 1]^T$$

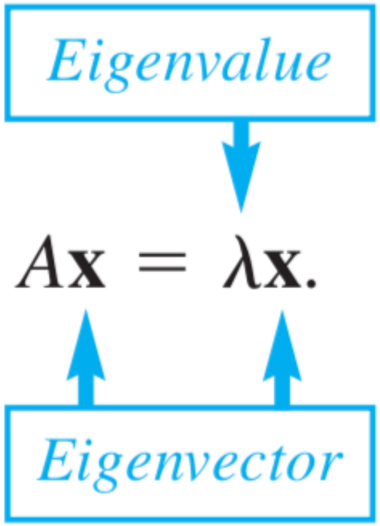
These points on the Z axis are EIGENVECTORS OF A With EIGENVALUE 1

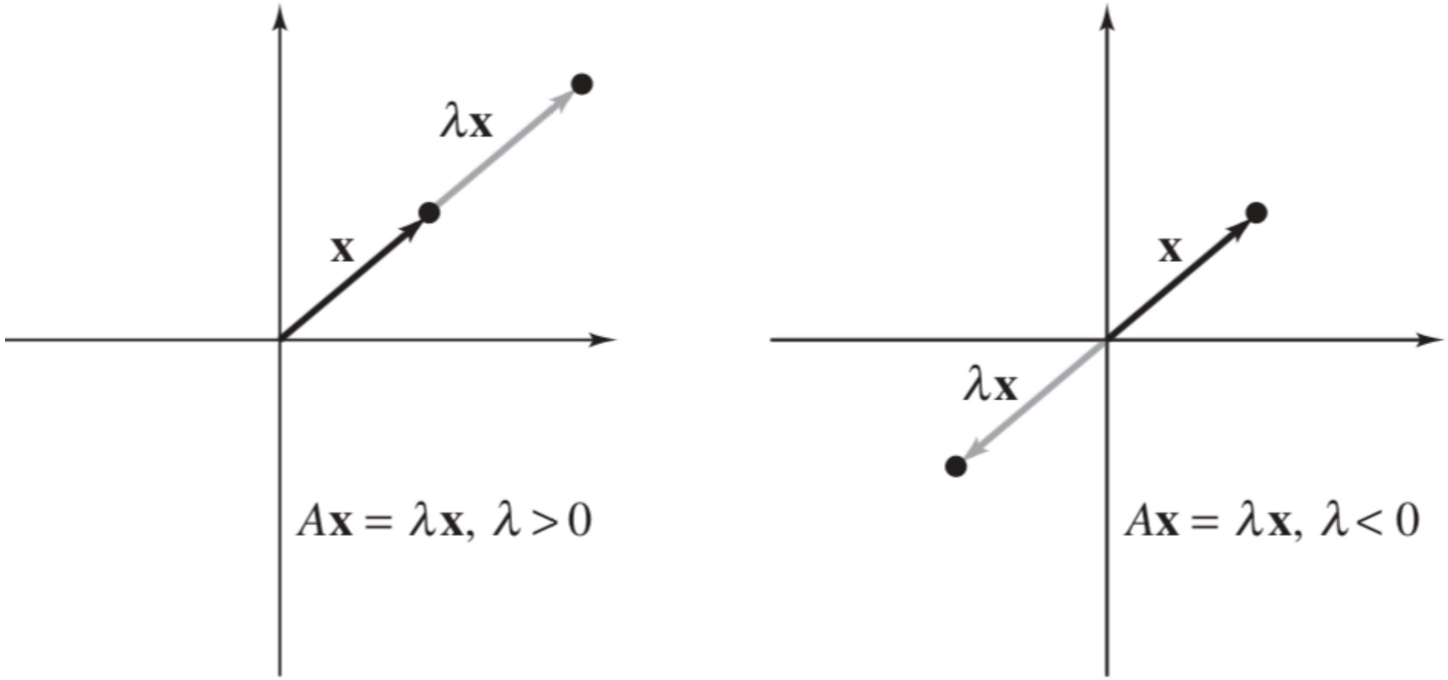
Intuition, Finding eigenpairs, Eigenspaces,Matrix similarity and diagonalization, Spectral theorem, Applications:

Eigenvectors of a Linear Transformation

If they exist their image is PARALLEL to the pre-image

$$T_{n\times n}\mathbf{u}_{n\times 1} = T(\mathbf{u}) = \lambda \mathbf{u}$$





EIGENPAIR

UNIVERSAL CONCEPT: IF T IS DIFFERENTIATION ON $f : \mathbb{R} \to \mathbb{R}$ THEN e^x IS AN EIGENVECTOR

Intuition, Finding eigenpairs, Eigenspaces, Matrix similarity and diagonalization, Spectral theorem, Applications:

FINDING EIGENVALUES

FIND ROOTS OF THE CHARACTERISTIC POLYNOMIAL!

$$A\mathbf{x} = \lambda\mathbf{x} \quad \Rightarrow (A - \lambda I)\mathbf{x} = \mathbf{0}$$

Rationale and definitions

It is desirable that the homogeneous system has not only the trivial solution:

(a) If it has only the trivial solution ($A - \lambda I$ invertible), then we have $\mathbf{x} = \mathbf{0}$

(b) So it is desirable that the determinant $|A - \lambda I|$ is zero

(c) The polynomial $|A - \lambda I| = 0$ is the characteristic equation and its solutions are the (real) eigenvalues of A

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = 0$$

$$(a_{11} - \lambda)(a_{22} - \lambda) - a_{21}a_{12} = 0$$

$$a_{11}a_{22} - \lambda a_{11} - \lambda a_{22} + \lambda^2 - a_{21}a_{12} = 0$$

$$\lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{21}a_{12} = 0$$

$$B^2 - 4AC \geq 0 \Rightarrow \underbrace{(a_{11} - a_{22})^2}_{\geq 0} + \underbrace{4a_{21}a_{12}}_{\geq 0 \text{ if symmetric}} \geq 0$$

Intuition, Finding eigenpairs, Eigenspaces, Matrix similarity and diagonalization, Spectral theorem, Applications:

Example#1: Find the eigenvalues of $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\begin{vmatrix} -1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} = 0 \qquad (-1-\lambda)(1-\lambda) = -(1+\lambda)(1-\lambda) = -(1-\lambda^2) = 0$$

$$q_A(\lambda) = \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$$

Example#2: Find the eigenvalues of $B = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}$

$$\begin{vmatrix} -1-\lambda & -2 \\ 1 & 1-\lambda \end{vmatrix} = 0 \qquad (-1-\lambda)(1-\lambda) + 2 = -(1-\lambda^2) + 2 = 0$$

$$q_B(\lambda) = \lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$$

Example#3: Find the eigenvalues of $C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$$\begin{vmatrix} 1-\lambda & 2 \\ 3 & 4-\lambda \end{vmatrix} = 0 \qquad (1-\lambda)(4-\lambda) - 6 = 4 - \lambda - 4\lambda + \lambda^2 - 6 = 0$$

$$q_B(\lambda) = \lambda^2 - 5\lambda - 2 = 0 \Rightarrow \lambda = \frac{5 \pm \sqrt{25+8}}{2}, \lambda_1 = 5.37, \lambda_2 = -0.37$$

Intuition, Finding eigenpairs, Eigenspaces, Matrix similarity and diagonalization, Spectral theorem, Applications:

Example#4: Find the eigenvalues of

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

$$q_A(\lambda) = \begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0$$

$$\left[(2-\lambda)(3-\lambda)(2-\lambda) + 2\right] - \left[(3-\lambda) + 2(2-\lambda) + 2(2-\lambda)\right] = 0$$

$$q_A(\lambda) = -\lambda^3 + 7\lambda^2 - 11\lambda + 5 = 0$$

RUFFINI: divisors of 5

	-1	7	-11	5
1		-1	6	-5
	-1	6	-5	0
1		-1	5	
	-1	5	0	

$$q_A(\lambda) = (\lambda - 1)(-\lambda^2 + 6\lambda - 5)$$
$$= (\lambda - 1)^2(\lambda - 5)$$

ALGEBRAIC MULTIPLICITY

Intuition, Finding eigenpairs, Eigenspaces, Matrix similarity and diagonalization, Spectral theorem, Applications:

FINDING EIGENVECTORS

SOLUTIONS OF THE ASSOCIATED HOMOGENEOUS SYSTEM

$$A\mathbf{x} = \lambda\mathbf{x} \quad \Rightarrow (A - \lambda I)\mathbf{x} = \mathbf{0}$$

Rationale and definitions

Given an eigenvalue λ the corresponding eigenvector is a solution of the homogeneous system $(A - \lambda I)\mathbf{x} = \mathbf{0}$

(a) $\mathbf{x} \in$ Ker(T_λ), $T_\lambda = A - \lambda I$

(b) The dimension dim(Ker(T_λ)) is the geometric multiplicity.

(c) The multiplicity of λ is the algebraic multiplicity.

Example#1: Find the eigenvalues of $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\begin{vmatrix} -1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} = 0 \qquad (-1-\lambda)(1-\lambda) = -(1+\lambda)(1-\lambda) = -(1-\lambda^2) = 0$$

$$q_A(\lambda) = \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$$

Example#5: Find the eigenVECTORS of $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\begin{vmatrix} -1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} = 0 \qquad q_A(\lambda) = \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$$

$\lambda = 1$

$$T_\lambda = \begin{bmatrix} -1-1 & 0 \\ 0 & 1-1 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \mathbf{x} = \textbf{Span}\{t(0,1)\}$$

$\lambda = -1$

$$T_\lambda = \begin{bmatrix} -1+1 & 0 \\ 0 & 1+1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \mathbf{x} = \textbf{Span}\{t(1,0)\}$$

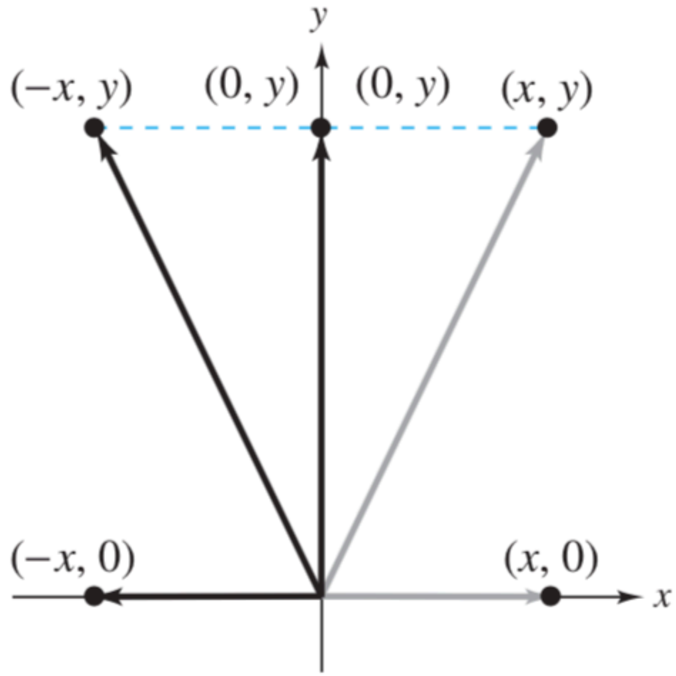
Intuition, Finding eigenpairs, Eigenspaces, Matrix similarity and diagonalization, Spectral theorem, Applications:

Example#5: Find the eigenVECTORS of $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

INTERPRETATION $(\lambda = 1, \mathbf{x} = (0,1))$ $(\lambda = -1, \mathbf{x} = (1,0))$

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ y \end{bmatrix} = (+1) \begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} = (-1) \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} -x \\ 0 \end{bmatrix}$$



A reflects vectors in the y-axis.

A IS A REFLECTION IN THE Y AXIS:
ONLY THE X AXIS AND THE Y AXES REMAIN DIRECTIONALLY INVARIANT

Intuition, Finding eigenpairs, Eigenspaces, Matrix similarity and diagonalization, Spectral theorem, Applications:

Example#4: Find the eigenvalues of

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

$$q_A(\lambda) = \begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0$$

$$\left[(2-\lambda)(3-\lambda)(2-\lambda)+2\right] - \left[(3-\lambda)+2(2-\lambda)+2(2-\lambda)\right] = 0$$

$$q_A(\lambda) = -\lambda^3 + 7\lambda^2 - 11\lambda + 5 = 0$$

RUFFINI: divisors of 5

	-1	7	-11	5
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	-1	6	-5	0
1		-1	5	
	-1	5	0	

$$q_A(\lambda) = (\lambda - 1)(-\lambda^2 + 6\lambda - 5)$$
$$= (\lambda - 1)^2(\lambda - 5)$$

ALGEBRAIC MULTIPLICITY

Intuition, Finding eigenpairs, Eigenspaces, Matrix similarity and diagonalization, Spectral theorem, Applications:

Example#4: Find the eigenVECTORS of $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

$$q_A(\lambda) = (\lambda - 1)(-\lambda^2 + 6\lambda - 5) = (\lambda - 1)^2(\lambda - 5)$$

$\lambda = 1$

$$T_{\lambda=1} = \begin{bmatrix} 2-1 & 2 & 1 \\ 1 & 3-1 & 1 \\ 1 & 2 & 2-1 \end{bmatrix} \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 2 & 1 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - R_1} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow$$

$$x_2 = \alpha$$

$$x_3 = \beta \quad \Rightarrow (x_1, x_2, x_3) = \textbf{Span}\{(-2\alpha - \beta, \alpha, \beta)\}$$

$$x_1 + 2\alpha + \beta = 0$$

$$x_1 = -2\alpha - \beta$$

$$\phi_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \phi_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow (x_1, x_2, x_3) = \textbf{Span}\{(-2\alpha - \beta, \alpha, \beta)\} = \textbf{Span}\{\alpha(-2,1,0) + \beta(-1,0,1)\}$$

Intuition, Finding eigenpairs, Eigenspaces, Matrix similarity and diagonalization, Spectral theorem, Applications:

Example#4: Find the eigenVECTORS of $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

$\lambda = 5$

$$q_A(\lambda) = (\lambda - 1)(-\lambda^2 + 6\lambda - 5) = (\lambda - 1)^2(\lambda - 5)$$

$$T_{\lambda=5} = \begin{bmatrix} 2-5 & 2 & 1 \\ 1 & 3-5 & 1 \\ 1 & 2 & 2-5 \end{bmatrix} \Rightarrow \begin{bmatrix} -3 & 2 & 1 & |0 \\ 1 & -2 & 1 & |0 \\ 1 & 2 & -3 & |0 \end{bmatrix} \xrightarrow{R_1 \leftarrow -\frac{1}{3}R_1} \begin{bmatrix} 1 & -\frac{2}{3} & -\frac{1}{3} & |0 \\ 1 & -2 & 1 & |0 \\ 1 & 2 & -3 & |0 \end{bmatrix}$$

$$\xrightarrow{R_2 \leftarrow R_2 - R_1} \begin{bmatrix} 1 & -\frac{2}{3} & -\frac{1}{3} & |0 \\ 0 & -\frac{4}{3} & \frac{4}{3} & |0 \\ 1 & 2 & -3 & |0 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 - R_1} \begin{bmatrix} 1 & -\frac{2}{3} & -\frac{1}{3} & |0 \\ 0 & -\frac{4}{3} & \frac{4}{3} & |0 \\ 0 & \frac{8}{3} & -\frac{8}{3} & |0 \end{bmatrix} \xrightarrow{R_2 \leftarrow -\frac{3}{4}R_2} \begin{bmatrix} 1 & -\frac{2}{3} & -\frac{1}{3} & |0 \\ 0 & 1 & -1 & |0 \\ 0 & \frac{8}{3} & -\frac{8}{3} & |0 \end{bmatrix}$$

$$\xrightarrow{R_3 \leftarrow \frac{3}{8}R_3} \begin{bmatrix} 1 & -\frac{2}{3} & -\frac{1}{3} & |0 \\ 0 & 1 & -1 & |0 \\ 0 & 1 & -1 & |0 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 - R_2} \begin{bmatrix} 1 & -\frac{2}{3} & -\frac{1}{3} & |0 \\ 0 & 1 & -1 & |0 \\ 0 & 0 & 0 & |0 \end{bmatrix}$$

$$x_3 = \alpha$$
$$\Rightarrow x_2 = x_3 = \alpha$$
$$x_1 = \alpha$$

$$\Rightarrow (x_1, x_2, x_3) = \textbf{Span}\{\alpha(1,1,1)\}$$

HOMEWORK OPTIONAL (Eigenvalues/vectors):

Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 1 & 1 \\ 4 & 2 & 0 \end{bmatrix}$$

SOLUTION:

1. Calculate the eigenvalues and their algebraic multiplicity

2. Calculate the eigenspace associated with each distinct eigenvalue (the base of the associated Ker) and also their geometric multiplicity.

3. Prove that all the eigenvectors obtained (together) are linear independent, but they do not form a base of $\mathbb{R}^3$

Intuition, Finding eigenpairs, Eigenspaces, Matrix similarity and diagonalization, Spectral theorem, Applications:

1. Calculate the eigenvalues and their algebraic multiplicity

$$q_A(\lambda) = \begin{vmatrix} 3-\lambda & 1 & -1 \\ 1 & 1-\lambda & 1 \\ 4 & 2 & 0-\lambda \end{vmatrix} = 0$$

$$\left[(3-\lambda)(1-\lambda)(-\lambda) + 4 - 2\right] - \left[(-1)(1-\lambda)4 + 2(3-\lambda) - \lambda\right] = 0$$

$$\left[(3-\lambda)(\lambda^2-\lambda) + 2\right] - \left[(4\lambda - 4) + 6 - 3\lambda\right] = 0$$

$$\left[3\lambda^2 - 3\lambda - \lambda^3 + \lambda^2 + 2\right] - \left[\lambda + 2\right] = 0$$

$$-\lambda^3 + 4\lambda^2 - 4\lambda = 0$$

$$q_A(\lambda) = \lambda^3 - 4\lambda^2 + 4\lambda = 0$$

$$= \lambda(\lambda^2 - 4\lambda + 4) = 0$$

$$\lambda^2 - 4\lambda + 4 = 0$$

$$\lambda = \frac{4 \pm \sqrt{16-16}}{2} = 2$$

Algebraic Multiplicities

$\lambda = 0$

$\lambda = 2$ **Double**

2. Calculate the eigenspace associated with each distinct eigenvalue (the base of the associated Ker) and also their geometric multiplicity.

$$q_A(\lambda) = \lambda(\lambda - 2)^2 = 0$$

$\lambda = 0$

$$T_0 = \begin{bmatrix} 3-0 & 1 & -1 \\ 1 & 1-0 & 1 \\ 4 & 2 & 0-0 \end{bmatrix} = 0 \Rightarrow \left[\begin{array}{ccc|c} 3 & 1 & -1 & 0 \\ 1 & 1 & 1 & 0 \\ 4 & 2 & 0 & 0 \end{array}\right] \xrightarrow{R_3 \to \frac{1}{3}R_1} \left[\begin{array}{ccc|c} 1 & \frac{1}{3} & -\frac{1}{3} & 0 \\ 1 & 1 & 1 & 0 \\ 4 & 2 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|c} 1 & \frac{1}{3} & -\frac{1}{3} & 0 \\ 0 & \frac{2}{3} & \frac{4}{3} & 0 \\ 4 & 2 & 0 & 0 \end{array}\right] \xrightarrow{R_3 \to R_3 - 4R_1} \left[\begin{array}{ccc|c} 1 & \frac{1}{3} & -\frac{1}{3} & 0 \\ 0 & \frac{2}{3} & \frac{4}{3} & 0 \\ 0 & \frac{2}{3} & \frac{4}{3} & 0 \end{array}\right] \xrightarrow{R_2 \to \frac{3}{2}R_2}$$

Intuition, Finding eigenpairs, Eigenspaces, Matrix similarity and diagonalization, Spectral theorem, Applications:

2. Calculate the eigenspace associated with each distinct eigenvalue (the base of the associated Ker) and also their geometric multiplicity.

$$q_A(\lambda) = \lambda(\lambda - 2)^2 = 0$$

$\lambda = 0$

$$\begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} & |0 \\ 0 & \frac{2}{3} & \frac{4}{3} & |0 \\ 0 & \frac{2}{3} & \frac{4}{3} & |0 \end{bmatrix} \xrightarrow{R_2 \to \frac{3}{2}R_2} \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} & |0 \\ 0 & 1 & 2 & |0 \\ 0 & \frac{2}{3} & \frac{4}{3} & |0 \end{bmatrix} \xrightarrow{R_3 \to R_3 - \frac{2}{3}R_2}$$

Geometric multiplicity =1 $\phi_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} & |0 \\ 0 & 1 & 2 & |0 \\ 0 & 0 & 0 & |0 \end{bmatrix} \Rightarrow \begin{aligned} x_3 &= \alpha \\ x_2 &= -2\alpha \\ x_1 &= \frac{1}{3}2\alpha + \frac{1}{3}\alpha = \alpha \end{aligned}$$

$\Rightarrow (x_1, x_2, x_3) =$ **Span**$\{(\alpha, -2\alpha, \alpha)\}$

Intuition, Finding eigenpairs, Eigenspaces, Matrix similarity and diagonalization, Spectral theorem, Applications:

2. Calculate the eigenspace associated with each distinct eigenvalue (the base of the associated Ker) and also their geometric multiplicity.

$$q_A(\lambda) = \lambda(\lambda - 2)^2 = 0$$

$\lambda = 2$

$$T_2 = \begin{bmatrix} 3-2 & 1 & -1 \\ 1 & 1-2 & 1 \\ 4 & 2 & 0-2 \end{bmatrix} = 0 \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 1 & -1 & 1 & 0 \\ 4 & 2 & -2 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -2 & 2 & 0 \\ 4 & 2 & -2 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - 4R_1} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & -2 & 2 & 0 \end{array}\right] \xrightarrow{R_2 \to -\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -2 & 2 & 0 \end{array}\right] \xrightarrow{R_3 \to R_3 + 2R_2} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \begin{array}{l} x_3 = \alpha \\ x_2 = \alpha \\ x_1 = 0 \end{array} \Rightarrow (x_1, x_2, x_3) = \textbf{Span}\{\alpha(0,1,1)\}$$

Geometric multiplicity =1

$$\phi_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Intuition, Finding eigenpairs, Eigenspaces, Matrix similarity and diagonalization, Spectral theorem, Applications:

3. Prove that all the eigenvectors obtained (together) are linear independent, but they do not form a base of $\mathbb{R}^3$

Eigenvectors $\phi_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ $\phi_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ **Are they LI?**

$$c_1 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow c_1 = c_2 = 0$$

$$\begin{bmatrix} 1 & 0 \\ -2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & |0 \\ -2 & 1 & |0 \\ 1 & 1 & |0 \end{bmatrix} \xrightarrow{R_2 \to R_2 + 2R_1} \begin{bmatrix} 1 & 0 & |0 \\ 0 & 1 & |0 \\ 1 & 1 & |0 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - R_1} \begin{bmatrix} 1 & 0 & |0 \\ 0 & 1 & |0 \\ 0 & 1 & |0 \end{bmatrix} \Rightarrow \begin{matrix} c_1 = 0 \\ c_2 = 0 \end{matrix}$$

$$c_1 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} x \neq 0 \\ y \neq 0 \\ z = 0 \end{bmatrix} \text{ ?}$$

CANNOT GENERATE THE FULL SPACE; NOT A BASE

Intuition, Finding eigenpairs, Eigenspaces, Matrix similarity and diagonalization, Spectral theorem, Applications:

MATRIX SIMILARITY

SIMILAR MATRICES HAVE THE SAME STRUCTURE

Rationale and definitions

Given two matrices A, B of the same size they are similar if exists an invertible matrix P so that $B = P^{-1}AP$. Then

(a) They have the same characteristic polynomial $q_A(\lambda) = q_B(\lambda)$ (i.e. they have the same spectrum)

(b) They have the same determinant and trace.

(c) They have the same rank.

(d) If the matrix B is diagonal and $B = P^{-1}AP$ then A is diagonalizable

$$|B - \lambda I| = |P^{-1}AP - \lambda I| = |P^{-1}AP - P^{-1}\lambda IP| = |P^{-1}(A - \lambda I)P|$$

$$= |P^{-1}| \cdot |(A - \lambda I)| \cdot |P| = |P^{-1}| \cdot |P| \cdot |(A - \lambda I)|$$

$$= |P^{-1}P| \cdot |(A - \lambda I)| = |A - \lambda I|$$

Intuition, Finding eigenpairs, Eigenspaces, Matrix similarity and diagonalization, Spectral theorem, Applications:

DIAGONALIZABLE MATRICES

DIAGONALIZABLE MATRICES ARE SIMILAR TO A DIAGONAL MATRIX

Rationale and definitions

A square matrix A is diagonalizable if it is similar to a diagonal matrix D, i.e. if $D = P^{-1}AP$.

(a) This is equivalent to check $PD = AP$ for some invertible matrix P

(b) An $n \times n$ matrix A is diagonalizable if its eigenvectors form a base of $\mathbb{R}^n$

(c) To diagonalize a matrix A set:

$$P = [\phi_1, \phi_2, \ldots, \phi_n], D = \textbf{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$$

Example#5: Is the following matrix diagonalizable?

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

$$q_A(\lambda) = \lambda(\lambda - 2)^2 = 0 \qquad \phi_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \qquad \phi_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

NO, BECAUSE THESE VECTORS DO NOT FORM A BASIS OF $\mathbb{R}^3$

Intuition, Finding eigenpairs, Eigenspaces, Matrix similarity and diagonalization, Spectral theorem, Applications:

Example#6: DIAGONALIZE (if possible) $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

$$q_A(\lambda) = (\lambda - 1)(-\lambda^2 + 6\lambda - 5) = (\lambda - 1)^2(\lambda - 5)$$

$\lambda = 1$

$$\phi_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \phi_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$\lambda = 5$

$$\phi_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} -2 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$P^{-1}AP = D$ **Check:** $PD = AP$

$$PD = \begin{bmatrix} -2 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} -2 & -1 & 5 \\ 1 & 0 & 5 \\ 0 & 1 & 5 \end{bmatrix} \quad AP = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} -2 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & -1 & 5 \\ 1 & 0 & 5 \\ 0 & 1 & 5 \end{bmatrix}$$

Intuition, Finding eigenpairs, Eigenspaces, Matrix similarity and diagonalization, Spectral theorem, Applications:

SPECTRAL THEOREM

ALL SYMMETRIC MATRICES ARE DIAGONALIZABLE!

Theorem and implications

A symmetric $n \times n$ matrix A is always diagonalizable as $D = P^{-1}AP$

where $P = [\phi_1, \phi_2, \ldots, \phi_n], D =$ diag$(\lambda_1, \lambda_2, \ldots, \lambda_n)$. Thus $A = \sum_{i=1}^{n} \lambda_i \tilde{\Phi}_i \tilde{\Phi}_i^T$

NORMALIZED

Because…

(a) It has exactly n LI eigenvectors (geometric multiplicities add one)

(b) These eigenvectors are also orthogonal and thus $P^{-1} = P^T$

(c) All the above is a consequence of the symmetry!

Example#6: Diagonalize the following symmetric matrix $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

Show that $A = \Phi \Lambda \Phi^T, \Lambda =$ diag$(\lambda_1, \lambda_2, \lambda_3)$

Intuition, Finding eigenpairs, Eigenspaces, Matrix similarity and diagonalization, Spectral theorem, Applications:

Example#6: Diagonalize the following symmetric matrix $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

Show that $A = \tilde{\Phi}\Lambda\tilde{\Phi}^T, \Lambda =$ **diag**$(\lambda_1, \lambda_2, \lambda_3)$

$$q_A(\lambda) = \begin{vmatrix} 3-\lambda & 0 & 0 \\ 0 & 2-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$\left[(3-\lambda)(2-\lambda)(2-\lambda)+0\right] - \left[0+0+(3-\lambda)\right] = 0$$

$$\left[(3-\lambda)(2-\lambda)(2-\lambda) - (3-\lambda)\right] = 0$$

$$(3-\lambda)\left[(2-\lambda)^2 - 1\right] = 0$$

$$(3-\lambda)\left[(4+\lambda^2-4\lambda)-1\right] = (3-\lambda)\left[\lambda^2-4\lambda+3\right] = 0$$

$$(3-\lambda)\left[(4+\lambda^2-4\lambda)-1\right] = (3-\lambda)\left[\lambda^2-4\lambda+3\right] = 0$$ $\lambda_1 = 3$

$$\lambda = \frac{4 \pm \sqrt{16-12}}{2} = \frac{4 \pm 2}{2} = 3{,}1 \qquad \lambda_2 = 3, \lambda_3 = 1$$

Intuition, Finding eigenpairs, Eigenspaces, Matrix similarity and diagonalization, Spectral theorem, Applications:

Example#6: Diagonalize the following symmetric matrix $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

$\lambda = 3$

$$q_A(\lambda) = (\lambda - 3)^2(\lambda - 1)$$

$$T_3 = \begin{bmatrix} 3-3 & 0 & 0 \\ 0 & 2-3 & -1 \\ 0 & -1 & 2-3 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} 0 & 0 & 0 & |0 \\ 0 & -1 & -1 & |0 \\ 0 & -1 & -1 & |0 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 0 & -1 & -1 & |0 \\ 0 & -1 & -1 & |0 \\ 0 & 0 & 0 & |0 \end{bmatrix}$$

$$\xrightarrow{R_1 \rightarrow -R_1} \begin{bmatrix} 0 & 1 & 1 & |0 \\ 0 & -1 & -1 & |0 \\ 0 & 0 & 0 & |0 \end{bmatrix} \xrightarrow{R_2 \rightarrow R_2 + R_1} \begin{bmatrix} 0 & 1 & 1 & |0 \\ 0 & 0 & 0 & |0 \\ 0 & 0 & 0 & |0 \end{bmatrix} \Rightarrow \begin{matrix} x_3 = \alpha \\ x_2 = -\alpha \\ x_1 = \beta \end{matrix}$$

$$\Rightarrow (x_1, x_2, x_3) = \textbf{Span}\{(\beta, -\alpha, \alpha)\} = \textbf{Span}\{\alpha(0,-1,1) + \beta(1,0,0)\}$$

$$\phi_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \phi_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Intuition, Finding eigenpairs, Eigenspaces, Matrix similarity and diagonalization, Spectral theorem, Applications:

Example#6: Diagonalize the following symmetric matrix $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

$\lambda = 1$

$$q_A(\lambda) = (\lambda - 3)^2(\lambda - 1)$$

$$T_1 = \begin{bmatrix} 3-1 & 0 & 0 \\ 0 & 2-1 & -1 \\ 0 & -1 & 2-1 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} 2 & 0 & 0 & |0 \\ 0 & 1 & -1 & |0 \\ 0 & -1 & 1 & |0 \end{bmatrix} \xrightarrow{R_1 \to \frac{1}{2}R_1} \begin{bmatrix} 1 & 0 & 0 & |0 \\ 0 & 1 & -1 & |0 \\ 0 & -1 & 1 & |0 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 + R_2} \begin{bmatrix} 1 & 0 & 0 & |0 \\ 0 & 1 & -1 & |0 \\ 0 & 0 & 0 & |0 \end{bmatrix} \Rightarrow \begin{matrix} x_3 = \alpha \\ x_2 = \alpha \\ x_1 = 0 \end{matrix}$$

$$(x_1, x_2, x_3) = \textbf{Span}\{(0, \alpha, \alpha)\} = \textbf{Span}\{\alpha(0,1,1)\}$$

$$\phi_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \phi_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \phi_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Intuition, Finding eigenpairs, Eigenspaces, Matrix similarity and diagonalization, Spectral theorem, Applications:

Example#6: Diagonalize the following symmetric matrix $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

Show that $A = \tilde{\Phi}\Lambda\tilde{\Phi}^T, \Lambda =$ **diag**$(\lambda_1, \lambda_2, \lambda_3)$

$$\lambda_1 = 3 \;\; \lambda_2 = 3, \lambda_3 = 1$$

$$\phi_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \phi_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \phi_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\phi_1 = \begin{bmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \phi_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \phi_3 = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Intuition, Finding eigenpairs, Eigenspaces, Matrix similarity and diagonalization, Spectral theorem, Applications:

Example#6: Diagonalize the following symmetric matrix $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

Show that $A = \tilde{\Phi}\Lambda\tilde{\Phi}^T, \Lambda = \textbf{diag}(\lambda_1, \lambda_2, \lambda_3)$

$$\lambda_1 = 3 \;\; \lambda_2 = 3, \lambda_3 = 1$$

$$\phi_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \phi_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \phi_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\phi_1 = \begin{bmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \phi_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \phi_3 = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

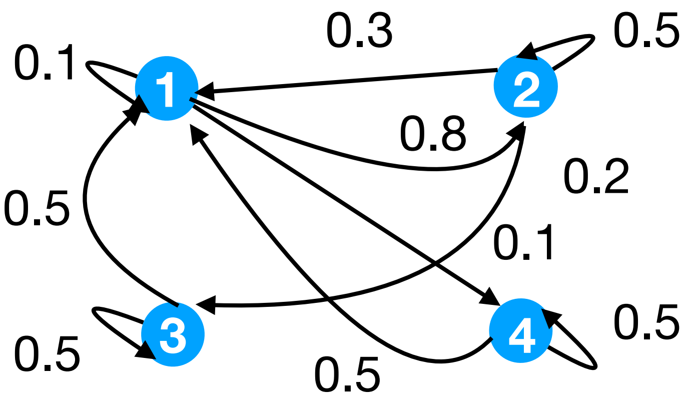
$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 & -\frac{3}{\sqrt{2}} & \frac{3}{\sqrt{2}} \\ 3 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Intuition, Finding eigenpairs, Eigenspaces, Matrix similarity and diagonalization, Spectral theorem, Applications:

Applications to graphs

	Page1	Page2	Page3	Page4
Page1	0.1	0.3	0.5	0.5
Page2	0.8	0.5	0	0
Page3	0	0.2	0.5	0
Page4	0.1	0	0	0.5

$$P = \begin{bmatrix} 0.1 & 0.3 & 0.5 & 0.5 \\ 0.8 & 0.5 & 0 & 0 \\ 0 & 0.2 & 0.5 & 0 \\ 0.1 & 0 & 0 & 0.5 \end{bmatrix}$$



$$P = \begin{bmatrix} 0.1 & 0.3 & 0.5 & 0.5 \\ 0.8 & 0.5 & 0 & 0 \\ 0.0 & 0.2 & 0.5 & 0 \\ 0.1 & 0 & 0 & 0.5 \end{bmatrix}, \mathbf{x}^0 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Where will be after many iterations?

$$\mathbf{x}^t = P^t \mathbf{x}^0$$

$$\mathbf{x} = P\mathbf{x}$$

Equilibrium condition

$$\mathbf{x} = \begin{bmatrix} x_1 = 0.5 \\ x_2 = 0.8 \\ x_3 = 0.3 \\ x_4 = 0.01 \end{bmatrix}$$

$\mathbf{x}$ is the EIGENVECTOR of $\lambda = 1$