Logic Slides

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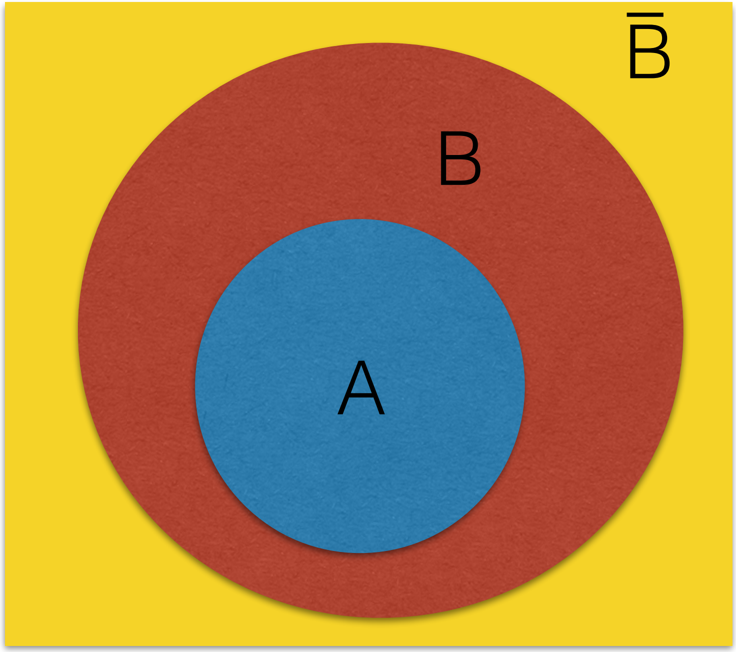
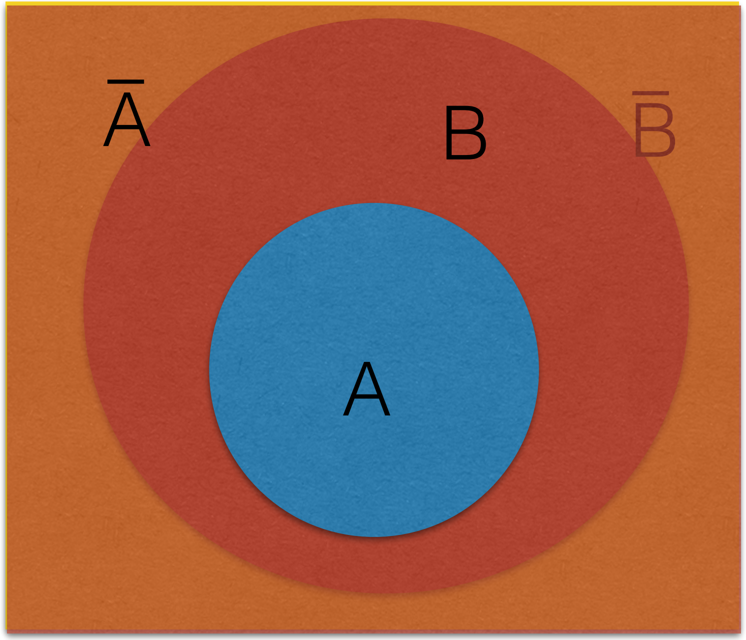
Implication & sets

- **if** A **then** B
  [logical/causal implication]
- A **suffices for** B, B **necessary for** A
  [A is a sufficient condition for B]
  [B is a necessary condition for A]
- **Not** A **or** B = (**not** (A **and not** B))
  [Either A does not happen or B does]
  [Cannot happen (A and not B)]
- A **only if** B = **only if** B **then** A
  [A can only happen if B does]
- **not** A **unless** B
  [not A can only happen within B, but not outside B so that we claim an implication]

$$A \Rightarrow B \equiv A \subseteq B$$



$$A \Rightarrow B \equiv \neg A \vee B \equiv \bar{A} \cup B$$

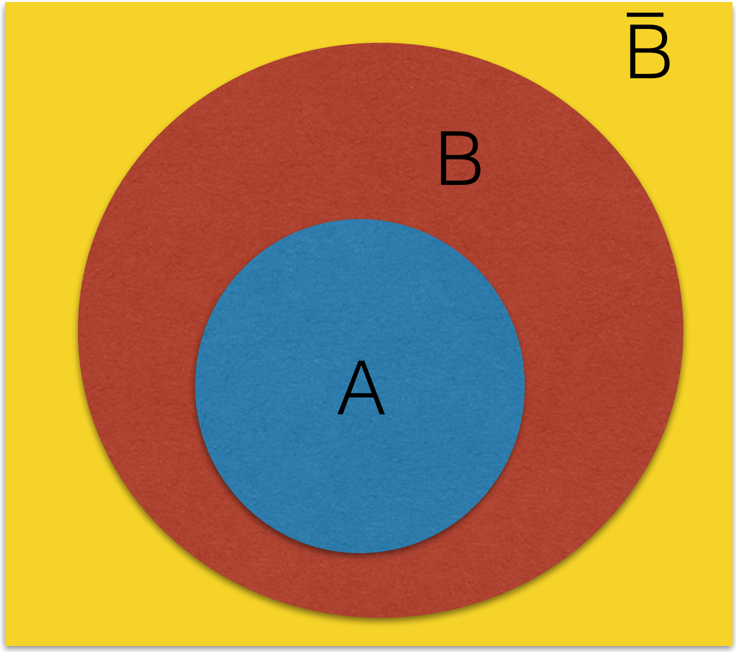


Examples

- **Conceptual framework:**
  CF = {sg=sing, da=dance,tp=tap}

- **P1: I dance and tap**
  P1: $sg \wedge da$
- **P2: I sing but I don't dance or tap**
  P2: $sg \wedge \neg(da \vee tp)$
- **P3: I neither sing nor dance**
  P3: $\neg sg \wedge \neg da \equiv \neg(sg \vee da)$
- **P4: It is true that I sing although it is not that I dance or tap**
  P4: $sg \wedge \neg(da \vee tp)$
- **P5: I either sing or dance, or it is false that I sing and tap**
  P5: $(sg \vee da) \vee \neg(sg \wedge tp)$
- **P6: It suffices that I sing so that I dance and don't tap**
  P6: $sg \Rightarrow da \wedge \neg tp$
- **P7: It is necessary that I sing so that I dance but don't tap**
  P7: $da \wedge \neg tp \Rightarrow sg$

$$A \Rightarrow B \equiv A \subseteq B$$



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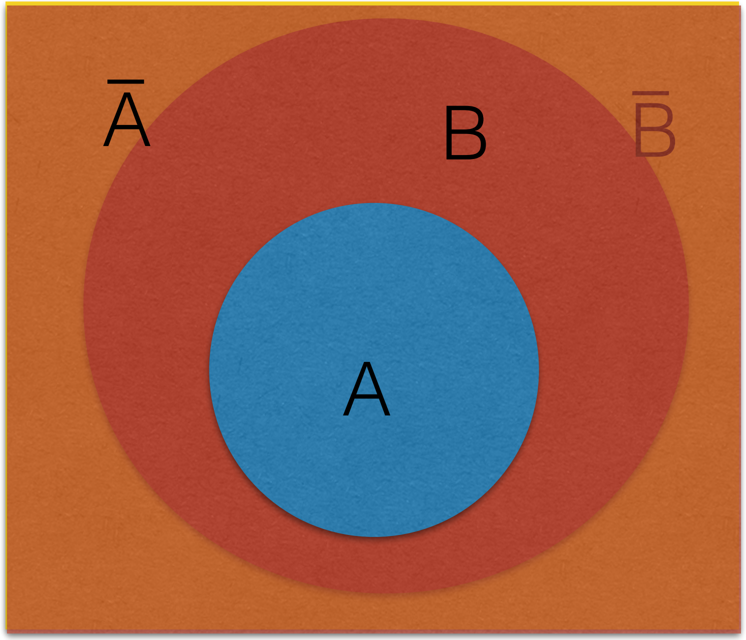
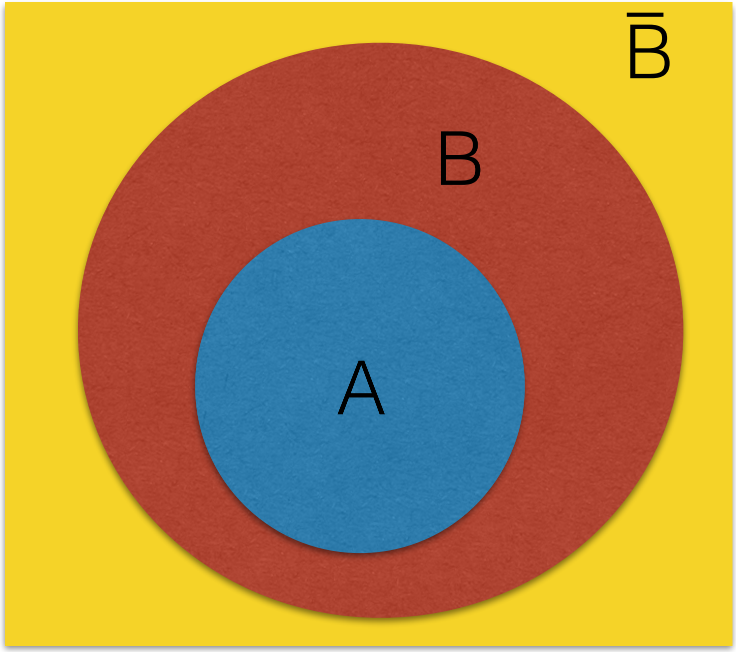


Examples

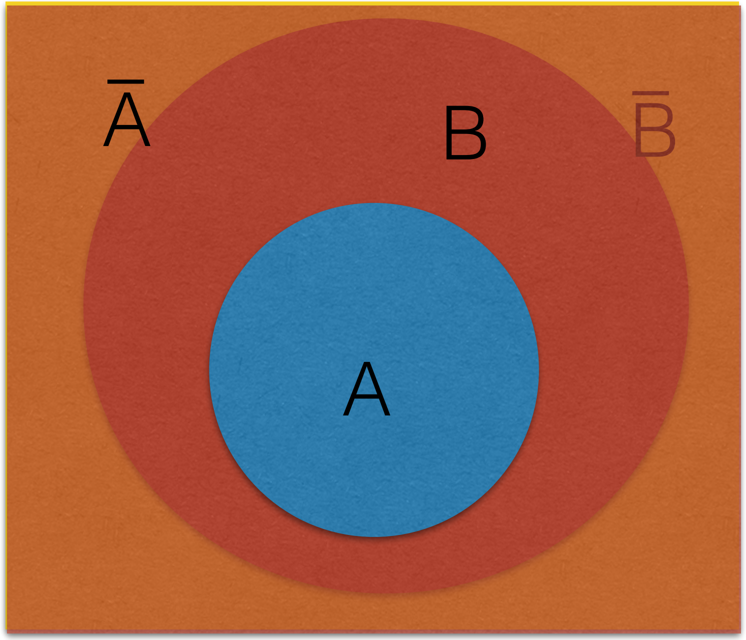
- **Conceptual framework:**
  CF = {sg=sing, da=dance,tp=tap}

- **P8: Only if I sign and dance I don't tap**
  P8: $\neg tp \Rightarrow sg \land da$
- **P9: It is both necessary and sufficient that I sing to dance**
  P9: $sg \iff da$
- **P10: I don't sing unless I dance and tap**
  P10: $sg \Rightarrow tp \lor da$
- **P11: I tap and dance unless I don't sing**
  P11: $\neg(tp \land da) \Rightarrow \neg sg$
- **P12: I neither sing nor dance unless I don't tap**
  P12: $\neg(\neg sg \land \neg da) \Rightarrow \neg tp$
- **P13: It is not necessary that I neither sing nor dance so that I tap**
  P13: $\neg(tp \Rightarrow sg \land da)$
- **P14: It is necessary, but not sufficient that I sing so that I dance**
  P14: $(da \Rightarrow sg) \land \neg(sg \Rightarrow da)$

$$A \Rightarrow B \equiv A \subseteq B$$



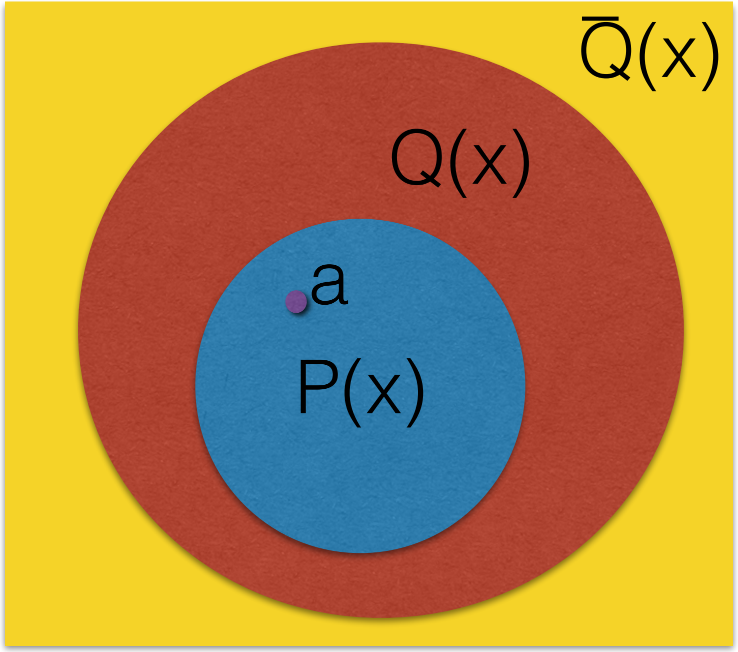
$$A \Rightarrow B \equiv \neg A \lor B \equiv \bar{A} \cup B$$



Relational Logic

- **P(x), Q(x,y)**
  Properties (actions, qualities), and relations between objects x,y. These objects belong to sets P, Q
- **P(Q(x),y)?: First-order-Logic (FoL)** cannot express predicates of predicates. Sets of sets is forbidden!
- **Quantifiers:** Exist and Forall
  X has property P = Exists x so that P is satisfied (set P is not empty)
- **Arity:**
  Number of arguments of a predicate.
- **Equivalences:** $\forall x : P(x) \equiv \neg\exists x : \neg P(x)$
  $\neg\forall x : P(x) \equiv \exists x : \neg P(x)$

$$\forall x : [P(x) \Rightarrow Q(x)]$$
$$\exists x : P(x) \wedge Q(x)$$

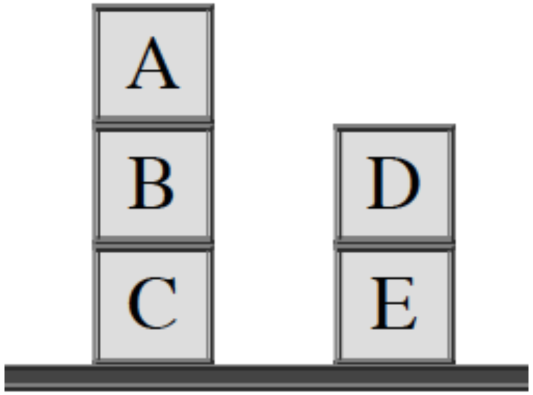


$$P(x) \Rightarrow (\forall y : Q(y)) \equiv \forall y : [P(x) \Rightarrow Q(y)]$$

Relational Logic: Example

- **Blocks world**
  Artificial Intelligence
- **Variables:** blocks {a,b,c,d,e}
- **Predicates:**
  on, above, stack, clear, table

$$\forall x : on(a,x) \Rightarrow above(a,x)$$
$$above(a,c) \wedge stack(a,x,c) \Rightarrow \exists x : on(a,x)$$
$$clear(x) \Rightarrow \neg\exists y : on(y,x)$$



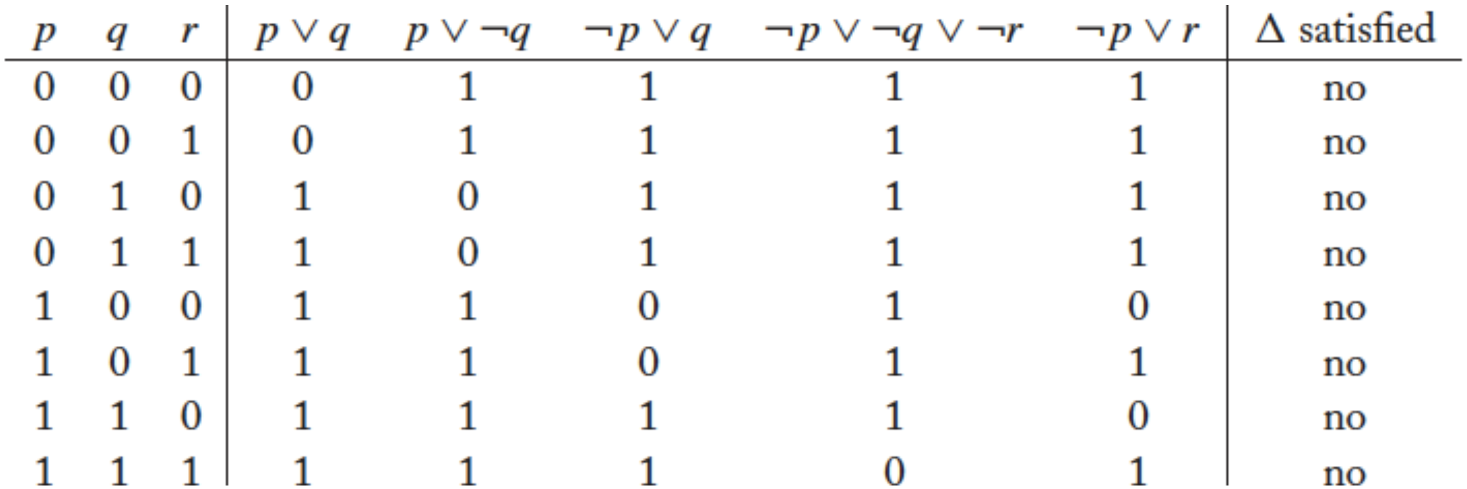
on(a,b)
on(b,c)
on(d,e)
above(a,c)
stack(a,b,c)
clear(x)?
table(c)
table(e)

Semantics

- **The Satisfiability Problem (SAT)**
  Given a propositional wff, the problem of determining whether it is satifiable, i.e. if it exists a **configuration** (set of truth assignments to the variables) for which the wff is true (model):

$$\Delta = \{p \vee q,\, p \vee \neg q,\, \neg p \vee q,\, \neg p \vee \neg q \vee \neg r,\, \neg p \vee r\}.$$

- **Thruth tables method:** look at all 2^n combinations:

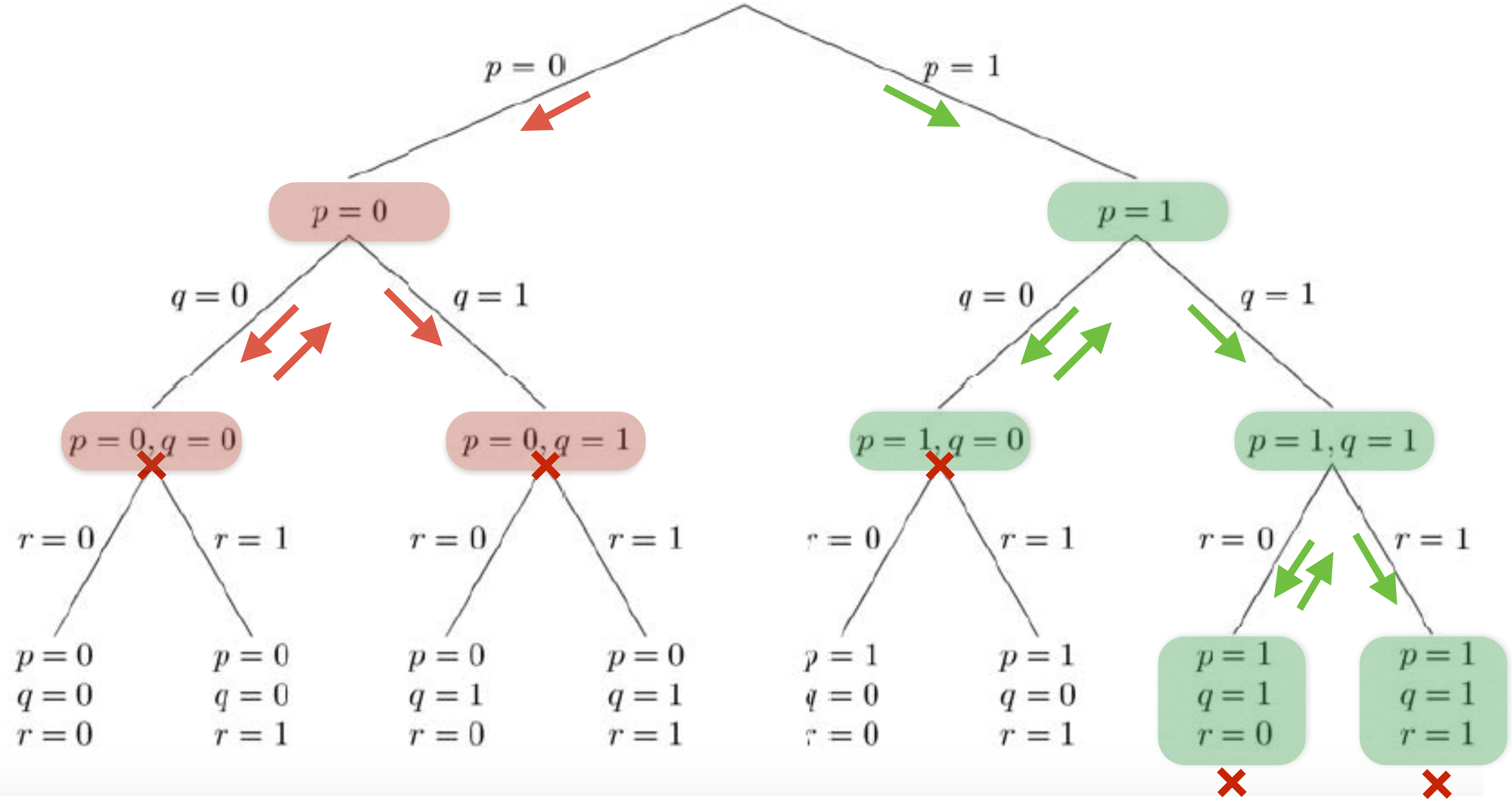


p	q	r	$p \vee q$	$p \vee \neg q$	$\neg p \vee q$	$\neg p \vee \neg q \vee \neg r$	$\neg p \vee r$	Δ satisfied
0	0	0	0	1	1	1	1	no
0	0	1	0	1	1	1	1	no
0	1	0	1	0	1	1	1	no
0	1	1	1	0	1	1	1	no
1	0	0	1	1	0	1	0	no
1	0	1	1	1	0	1	1	no
1	1	0	1	1	1	1	0	no
1	1	1	1	1	1	0	1	no

- **Taulology** = all model, **Contigency** = at least one model, **Contradiction** = none model (all counter-model).

Semantics: SAT “backtracking”

- **Backtracking O(2^n) in the worst case**



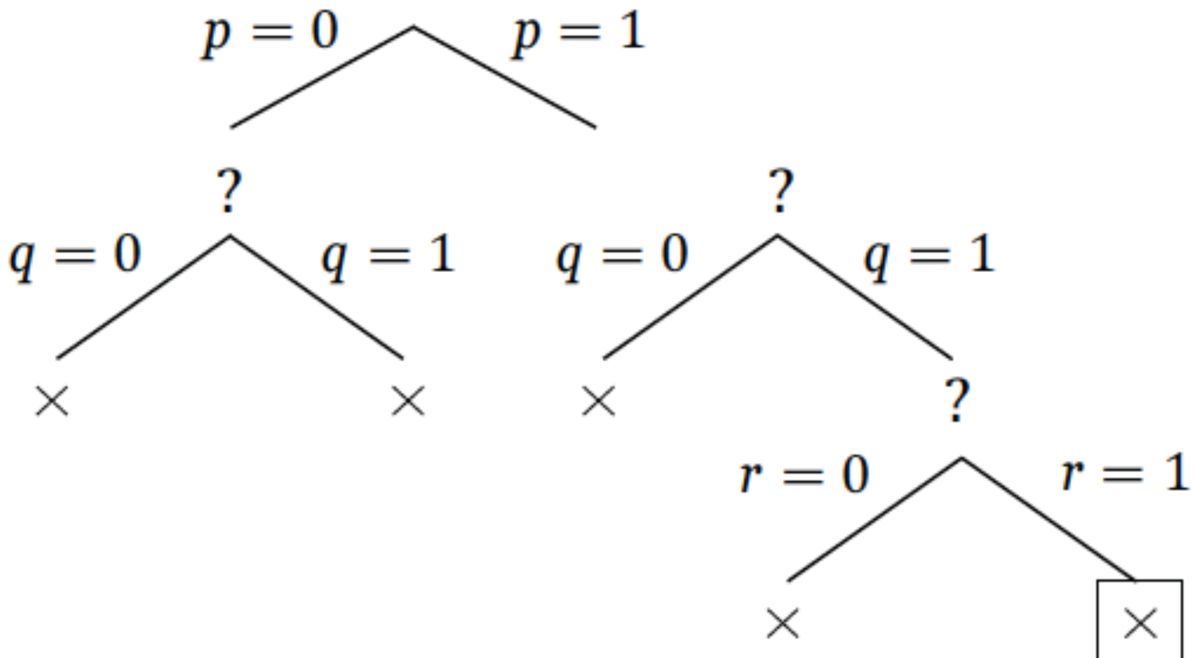
Semantics

$$\Delta = \{p \vee q,\ p \vee \neg q,\ \neg p \vee q,\ \neg p \vee \neg q \vee \neg r,\ \neg p \vee r\}.$$

p	q	r	$p \vee q$	$p \vee \neg q$	$\neg p \vee q$	$\neg p \vee \neg q \vee \neg r$	$\neg p \vee r$	Δ satisfied
0	0	0	0	1	1	1	1	no
0	0	1	0	1	1	1	1	no
0	1	0	1	0	1	1	1	no
0	1	1	1	0	1	1	1	no
1	0	0	1	1	0	1	0	no
1	0	1	1	1	0	1	1	no
1	1	0	1	1	1	1	0	no
1	1	1	1	1	1	0	1	no

Semantics: SAT “backtracking”

- **Backtracking O(2^n) in the worst case:**
  * Start but void at the root.
  * Make a partial assignment p=0
  * ?: ask for the following variable: q
  * if the wff is falsified then **cut** and close brach
  * if all leaves are found false then return false.
  * If at least one leave in the tree is true then one SAT
  * Checking a tautology requires looking at all 2^n leaves



Semantics: Refutation

- **The counter-example method:**
  * A reasoning R is **valid** if its associated **implication** is a **tautology**:

$$R : \{P_1, P_2, \ldots, P_n \models Q\} \equiv P_1, P_2, \ldots, P_n \Rightarrow Q$$

- **Method:**
  1. Assume R is false, i.e. $P_1, P_2, \ldots P_n$ are true and Q is false.
  2. From Q false, if the variables in Q are non-compatible with those of any in $P_1, P_2, \ldots P_n$ then claim **contradiction**.
  3. Contradiction means that R has no counter-examples, i.e. that it is a **tautology**!

$$R : \{P_1, P_2 \models Q\} \equiv P_1, P_2 \Rightarrow Q$$

$P_1 : p \Rightarrow q$
$P_2 : q \Rightarrow r$
$Q : p \Rightarrow r$

Q is false if $p = 1 \wedge r = 0$
P_1 is true if $p = 1 \wedge q = 1$
P_2 is true if $q = 1 \wedge r = 1$ but $r=0$

contradiction!

Semantics: Refutation

- **Exercise#7:**

$P_{1,2}$: **if** we lit the lamps A and B **then** we read
else we go to sleep

P_3: we go to sleep

Q: at least one lamp does not lit

Formalization:

$P1 : A \wedge B \Rightarrow R$

$P2 : \neg(A \wedge B) \Rightarrow S$

$P3 : S$

$Q : \neg A \vee \neg B$

Reasoning:

$$(A \wedge B \Rightarrow R) \wedge (\neg(A \wedge B) \Rightarrow S) \wedge S \Rightarrow \neg A \vee \neg B$$

Semantics: Refutation

- **Exercise#7:** Refutation:

$$\underbrace{(A \wedge B \Rightarrow R) \wedge (\neg(A \wedge B) \Rightarrow S) \wedge S}_{1} \Rightarrow \underbrace{\neg A \vee \neg B}_{0}$$

$\neg A \vee \neg B$	FALSE if **A**=1,**B**=1
S	TRUE if **S**=1
$(\neg(A \wedge B) \Rightarrow S)$	TRUE if **S**=1
$(A \wedge B \Rightarrow R)$	TRUE since **A**=1,**B**=1,**R**=1

There is NO CONTRADICTION!

Semantics: Refutation

- **Exercise#8:**

P$_{1,2}$**:** **if** we lit the lamps A and B **then** we read
else we go to sleep

P$_3$**:** at least one lamp does not lit

Q: we go to sleep

Formalization:

$P1 : A \wedge B \Rightarrow R$

$P2 : \neg(A \wedge B) \Rightarrow S$

$P3 : \neg A \vee \neg B$

$Q : S$

Reasoning:

$$(A \wedge B \Rightarrow R) \wedge (\neg(A \wedge B) \Rightarrow S) \wedge (\neg A \vee \neg B) \Rightarrow S$$

Semantics: Refutation

- **Exercise#8:**

Refutation:

$$\underbrace{(A \wedge B \Rightarrow R) \wedge (\neg(A \wedge B) \Rightarrow S) \wedge (\neg A \vee \neg B)}_{1} \Rightarrow \underbrace{S}_{0}$$

S FALSE if **S**=0

$(\neg(A \wedge B) \Rightarrow S)$ TRUE if **S**=0,**A**=1,**B**=1

$(\neg A \vee \neg B)$ TRUE if **A**=0 or **B**=0

But **A**=1 and **B**=1

CONTRADICTION!, so claim Tautology

Semantics: Refutation

- **Exercise#24:**

I go to the faculty or stay at home but not both things (**P_1**). For going to the faculty it is necessary that I have a class of maths (**P_2**) and for studying Logic it is enough that I stay at home (**P_3**). Therefore, I study Logic, since I don't go the the class of maths (**Q**).

Formalization:

CF = {**fa:** faculty, **ho:** home, **ma:** class of maths, **lo:** study Logic}

P1 : $(fa \lor ho) \land \neg(fa \land ho)$

P2 : $fa \Rightarrow ma$

P3 : $ho \Rightarrow lo$

Q : $\neg ma \Rightarrow lo$

$$\{[(fa \lor ho) \land \neg(fa \land ho)] \land (fa \Rightarrow ma) \land (ho \Rightarrow lo)\} \Rightarrow (\neg ma \Rightarrow lo)$$

Semantics: Refutation

- **Exercise#24:**

Refutation:

$$\underbrace{\{[(fa \lor ho) \land \neg(fa \land ho)] \land (fa \Rightarrow ma) \land (ho \Rightarrow lo)\}}_{1} \Rightarrow \underbrace{(\neg ma \Rightarrow lo)}_{0}$$

$(\neg ma \Rightarrow lo)$ FALSE if **ma**=0,**lo**=0

$(fa \Rightarrow ma)$ TRUE if **ma**=0,**fa**=0

$(ho \Rightarrow lo)$ TRUE if **lo**=0,**ho**=0

$\{[(fa \lor ho) \land \neg(fa \land ho)]$ TRUE (must be)

$\{[(0 \lor 0) \land \neg(0 \land 0)]$

FALSE! we have a contradiction!

Semantics: Refutation

- **Exercise#25:**

If the red ball hits the white ball, then the white ball moves (**P_1**). It is enough that the white ball moves to hit the green ball and win (**P_2**). The red ball is hit (**P_3**). You win (**Q**).

Formalization:

CF = {**rb:** red ball hit, **wb:** white ball moves, **gb:** green ball hit, **wi:** win}

P1 : $rb \Rightarrow wb$

P2 : $wb \Rightarrow (gb \wedge wi)$

P3 : rb

Q : wi

$$[(rb \Rightarrow wb) \wedge (wb \Rightarrow (gb \wedge wi)) \wedge rb] \Rightarrow wi$$

Semantics: Refutation

- **Exercise#25:**

Refutation:

$$\underset{1}{\underline{[(rb \Rightarrow wb) \land (wb \Rightarrow (gb \land wi)) \land rb]}} \Rightarrow \underset{0}{\underline{wi}}$$

wi FALSE if **wi**=0

(wb ⇒ (gb ∧ wi)) TRUE since **wi**=0, then **wb**=0

(rb ⇒ wb) TRUE if **wb**=0 and **rb**=0

rb r**b** must be 1, but it is 0

FALSE! we have a contradiction!

Natural Deduction

- **Rule-based inference**
  Given a set of premises (hypothesis), use allowed inference rules to obtain a conclusion.

$$R : \{P_1, P_2, \ldots, P_n \models Q\} \equiv P_1, P_2, \ldots, P_n \Rightarrow Q$$

- **Natural way of reasoning:** using valid inference rules.
  List hypotesis and use rules to reach Q.

1. P_1
2. P_2

…

n. P_n

Use premises as hypothesis (assumed to be true)

…

n+p….

Obtain P_i from P_j **using the rule** R_k

Eventually obtain Q

Natural Deduction

- **Strategies:** direct proof vs refutation.
  * Direct proof: start from hypothesis and add a new partial conclusion at any cycle until reaching Q.
  * Refutation: negate conclusion, add it as a hypothesis and look for a contradiction.

- Direct Proof: Example (**exercise#24**).
  P1 : (fa∨ho)∧¬(fa∧ho), P2 :fa⇒ma, P3 :ho⇒lo, Q : ¬ma ⇒ lo

1. (fa∨ho)∧¬(fa∧ho)
2. fa⇒ma
3. ho⇒lo

Assumption

4. ¬ma
5. ¬fa modus tollens ¬fa from 2,4: MT
6. fa∨ho eliminate conj in 1: EC
7. ho disjunctive silogism 5,6: SD
8. lo modus ponens 3,7: MP

Close

9. ¬ma ⇒ lo theorem of deduction 4,8: TD

Natural Deduction

- Refutation Proof: Example (**exercise#25**).

$$[(rb \Rightarrow wb) \land (wb \Rightarrow (gb \land wi)) \land rb] \Rightarrow wi$$

1. $rb \Rightarrow wb$	
2. $wb \Rightarrow (gb \land wi)$	
3. rb	
4. $\neg wi$	Negate conclusion
5. wb	modus ponens from 1,3: MT
6. $gb \land wi$	modus ponens from 2,5: MP
7. wi	eliminate disjunction 6: ED
8. $wi \land \neg wi$	Introduce conjunction 4,7: IC
9. wi	introduce negation 4: IN

Natural Deduction

P1 : $A \Rightarrow \neg B$
P2 : $\neg A \Rightarrow C$
P3 : $\neg E \Rightarrow F$
P4 : $E \Rightarrow D$
Q : $(B \Rightarrow C) \wedge (\neg F \Rightarrow D)$

- **Exercise#25**.

1. $A \Rightarrow \neg B$
2. $\neg A \Rightarrow C$
3. $\neg E \Rightarrow F$
4. $E \Rightarrow D$
   5. B (assume true first implication's head)
   6. $\neg A$, MT 1,5
   7. C, MP 2,6
   8. $B \Rightarrow C$, TD 5,7
   9. $\neg F$ (assume true second implication's head)
   10. E, MT 3,9
   11. D, MP 4,10
   12. $\neg F \Rightarrow D$, TD 10,11
13. $(B \Rightarrow C) \wedge (\neg F \Rightarrow D)$, IC 8,12

Natural Deduction

P1 :rb⇒mb
P2 :av⇒gp
P3 :rb∨av
Q : mb ∨ gp

- **Exercise#26**.

1. rb⇒mb
2. av⇒gp
3. rb ∨ av
4. ¬(mb ∨ gp) (refutation: negate Q)
    5. ¬mb ∧ ¬gp DeMorgan 4
    6. ¬gp EC 5
    7. ¬av MT 2,6
    8. rb SD 3,7
    9. mb MP 1,8
    10. ¬mb EC 5
    11. ¬mb ∧ mb IC 9,10
12. mb ∨ gp IN 4-11

Natural Deduction

P1 : $A \wedge B \Rightarrow L$
P2 :$\neg(A \wedge B) \Rightarrow D$
P3 :$\neg A \vee \neg B$
Q:D

- **Exercise#6**.

1. $A \wedge B \Rightarrow L$
2. $\neg(A \wedge B) \Rightarrow D$
3. $\neg A \vee \neg B$
4. $\neg D$ (refutation: negate Q)
    5. $\neg\neg(A \wedge B)$ MT 2,4
    6. $A \wedge B$ EN 5
    7. A EC 6
    8. $\neg B$ SD 3,7
    9. B EC 6
    10. $B \wedge \neg B$ IC 8,9
11. $\neg\neg D$ IN 4-10
12. D EN 11

Natural Deduction

P1 :$\neg(es \Rightarrow ap \wedge fe)$

P2 :$ap \wedge fe \Rightarrow es)$

P3 : $\neg es$

Q : $\neg ap \wedge \neg fe$

- **Exercise#18**.

1. $\neg(es \Rightarrow ap \wedge fe)$
2. $ap \wedge fe \Rightarrow es$
3. $\neg es$
4. $\neg(\neg ap \wedge \neg fe)$ (refutation: negate Q)
    5. $ap \vee fe$ DeMorgan,4
    6. $\neg(\neg es \vee (ap \wedge fe))$ Div,1
    7. $es \wedge \neg(ap \wedge fe))$ DeMorgan 6
    8. es EC 8
    9. $es \wedge \neg es$ IC 3,9
10. $\neg ap \wedge \neg fe$ ECQ 8