# Linear Algebra for Computer Science

## An incremental document:

## From Systems and Matrices to Eigenvalues and Eigenvectors

Francisco Escolano

**Notes based on the text**
**"Elementary Linear Algebra" by Larson and Falvo, 6th edition**

## Formulating & Solving Linear Systems

Linear vs non-Linear

- Gauss-Jordan
- Homogeneous Systems
- Least squares
- Curve fitting
- Traffic problems
- Dirichlet problems
- Neural Networks
- Echelon form

## Vectors & Matrices

### Matrices and Systems

- Properties of matrices
- Product
- Transpose
- Elementary Matrices
- Inverse of a matrix
- Application to Graphs

## Linear Transformations

Properties
Matr
Kernel
Isomorphism
Geometric
Isometry

**ONLY THE CONCEPT**

- Robo
- Vision
- Graphics

## Vector Spaces & Matrices

Polynomials Lines, planes, hyperplanes

Spaces and subspaces
Linear combinations
Bases and dimension
Change of basis

Rank & Nullity

- Dot & Cross products
- Norms and projections
- Stochastic matrices
- PCA

## Eigenvalues and Eigenvectors

Eigenvectors/values and transformations

Finding eigenpairs
Eigenspaces

Quadratic forms and their rotation
Similarity & Diagonalization
Graph characterization & PageRank

Matrix exponentiation
Systems of Differential equations

Solving an Homogeneous System per eigenvalue

For largest eigenvalue NO NEED OF system solving

$$A\mathbf{x} = \lambda\mathbf{x}$$

Each eigenvalue determines a subspace and the dimensions indicate whether A is diagonalizable

Eigenpairs allow both lossless and lossy changes of basis (PCA)

# Eigenvalues and Eigenvectors

**Eigenvectors/values and transformations**

**Finding eigenpairs**

**Eigenspaces**

**Quadratic forms and their rotation**

**Similarity & Diagonalization**

**Graph characterization & PageRank**

**Matrix exponentiation**

**Systems of Differential equations**

Eigenvectors and eigenvalues Define rotation matrices/axes In 2D and 3D

Symmetric Matrices have Real eigenvalues and are diagonalizable

Diagonalization enables new operations In matrices , e.g. expm,() logm(), sin(), some of them are useful in graphs

Spectra define the DNA of graphs & eigenvectors give the steady state of random walks

# 1. Formulating and solving linear systems

**Linearity vs non-linearity, examples of applications, Gauss and Gauss-Jordan reduction, Echelon form, Homogeneous systems :**

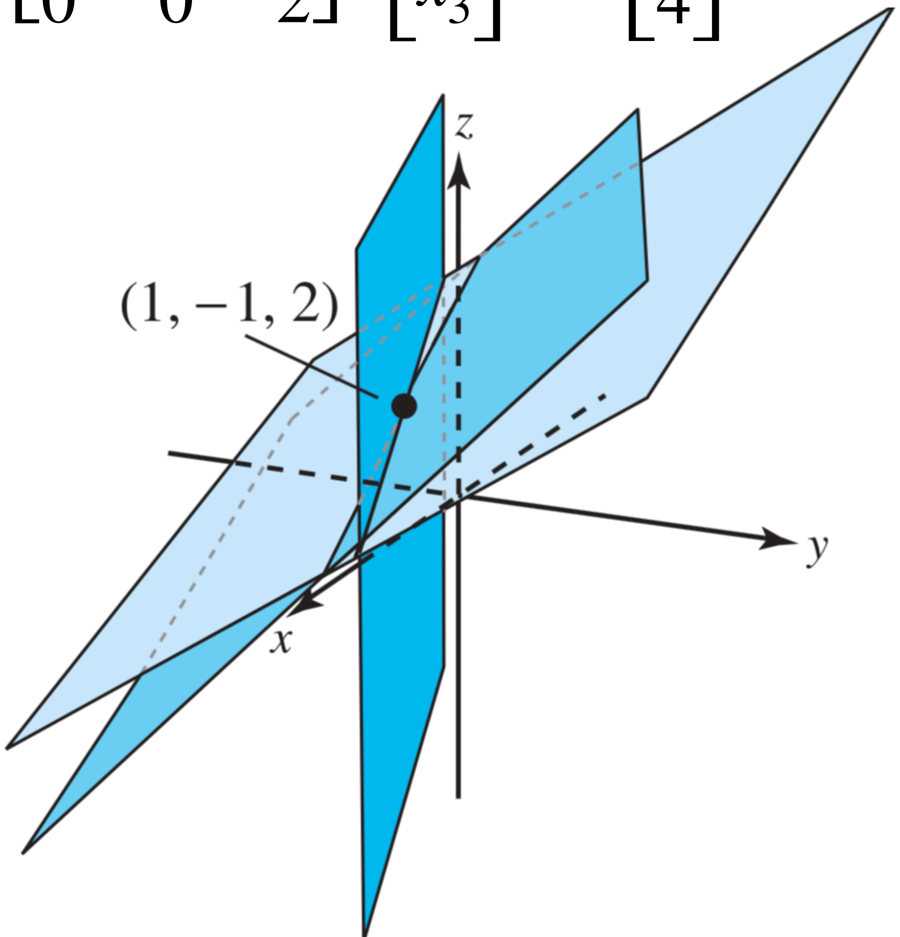
Francisco Escolano

## Linearity vs non-linearity, examples of applications, Gauss and Gauss-Jordan reduction, Echelon form, Homogeneous systems :

**Linear Systems are Geometric configurations**

$$A\mathbf{x} = \mathbf{b}$$

$$\begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \\ 4 \end{bmatrix}$$

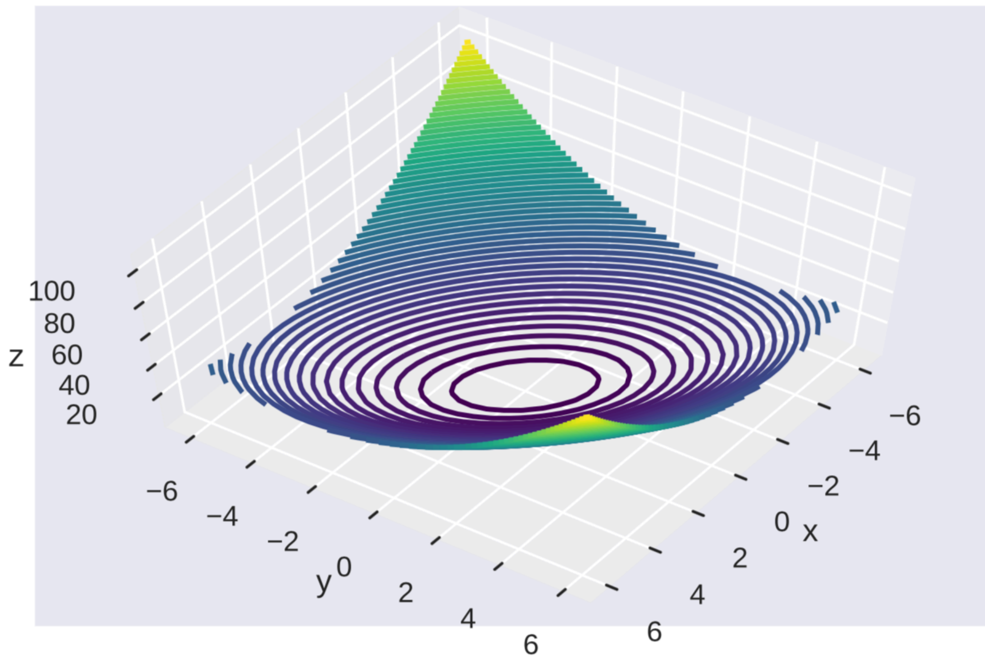



**Functions are QUADRATIC FORMS**

$$f(\mathbf{x}) = \mathbf{x}^T A \mathbf{x}$$

$$A = [x\ y] \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

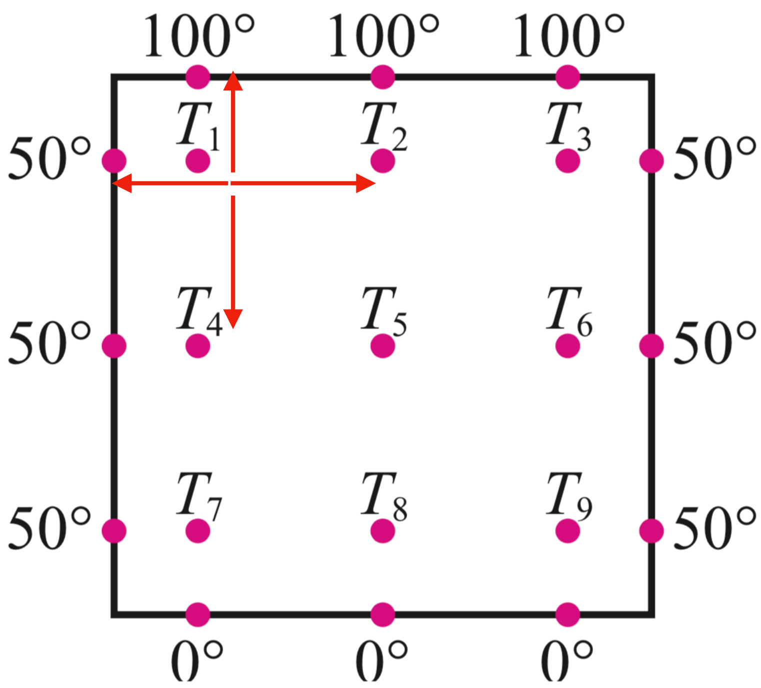
$$f(x, y) = x^2 + xy + y^2$$



# Linearity vs non-linearity, examples of applications, Gauss and Gauss-Jordan reduction, Echelon form, Homogeneous systems :

## Harmonic Analysis

**When the UNKNOWNS are the INNER NODES GIVEN THE KNOWN ONES**




**Harmonic (Linear) hypothesis:**

$$T_i = \frac{1}{|N_i|} \sum_{j \in N_i} T_j$$

**Find the Temperature in $T_i$ Given those of the border**

$$T_1 = \frac{1}{4}(50 + 100 + T_2 + T_4)$$
$$T_2 = \frac{1}{4}(T_1 + 100 + T_3 + T_5)$$
$$T_3 = \frac{1}{4}(T_2 + 100 + 50 + T_6)$$
$$T_4 = \frac{1}{4}(50 + T_1 + T_5 + T_7)$$
$$T_5 = \frac{1}{4}(T_4 + T_2 + T_6 + T_8)$$
$$T_6 = \frac{1}{4}(T_5 + T_3 + 50 + T_9)$$
$$T_7 = \frac{1}{4}(50 + T_4 + T_8 + 0)$$
$$T_8 = \frac{1}{4}(T_7 + T_5 + T_9 + 0)$$
$$T_9 = \frac{1}{4}(T_8 + T_6 + 50 + 0)$$

**Unique solution in this case. Why?**

## Linearity vs non-linearity, examples of applications, Gauss and Gauss-Jordan reduction, Echelon form, Homogeneous systems :

### Curve Fitting

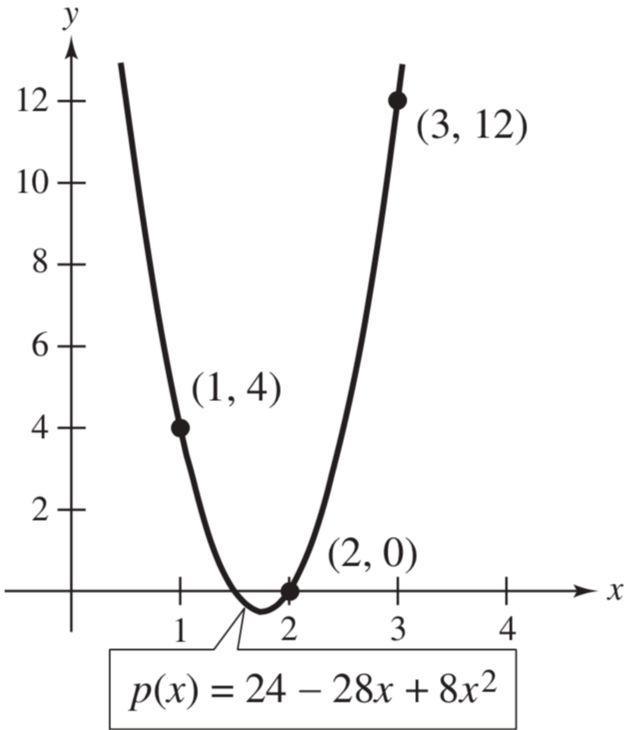
**When the UNKNOWNS are the COEFICIENTS OF A CURVE**

$$p(x) = a_0 + a_1 x + a_2 x^2$$

$$x_i, p(x_i) = \{(1,4), (2,0), (3,12)\}$$

$$\begin{aligned} p(x_1) &= a_0 + a_1 x_1 + a_2 x_1^2 \\ p(x_2) &= a_0 + a_1 x_2 + a_2 x_2^2 \\ p(x_3) &= a_0 + a_1 x_3 + a_2 x_3^2 \end{aligned}$$

$$\begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} p(x_1) \\ p(x_2) \\ p(x_3) \end{bmatrix}$$

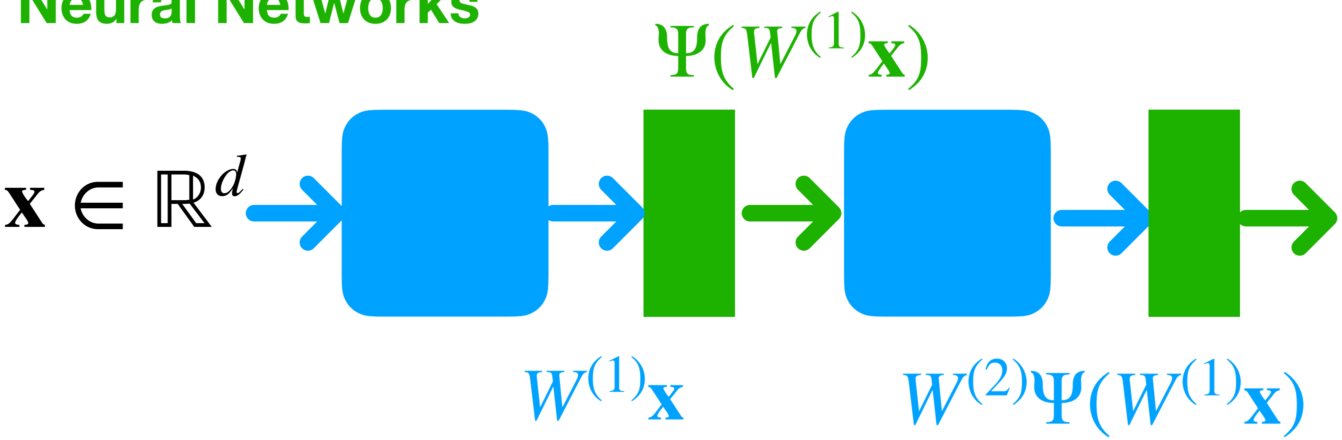



$$\begin{bmatrix} 1 & 1 & 1^2 \\ 1 & 2 & 2^2 \\ 1 & 3 & 3^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 12 \end{bmatrix}$$

**Solution=(24, -28, 8)**

# Linearity vs non-linearity, examples of applications, Gauss and Gauss-Jordan reduction, Echelon form, Homogeneous systems :

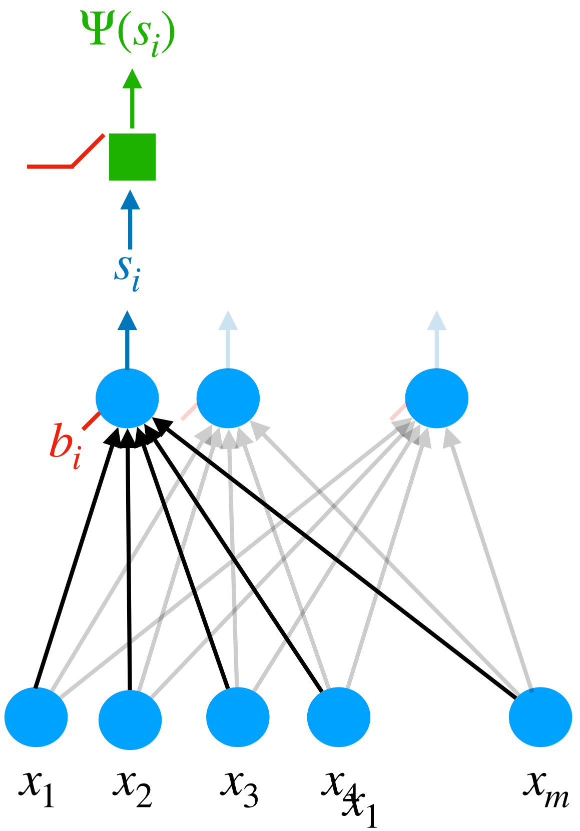
## Neural Networks




$\mathbf{z} \in \mathbb{R}^o$

$\mathbf{y} = \Psi(W^{(2)}\Psi(W^{(1)}\mathbf{x}))$

$Min_{W^{(1)},W^{(2)}} \|\mathbf{z} - \mathbf{y}\|^2$




$\Psi(S) = \Psi(W\mathbf{x})$

$\Psi(s_i) = max(0, s_i)$

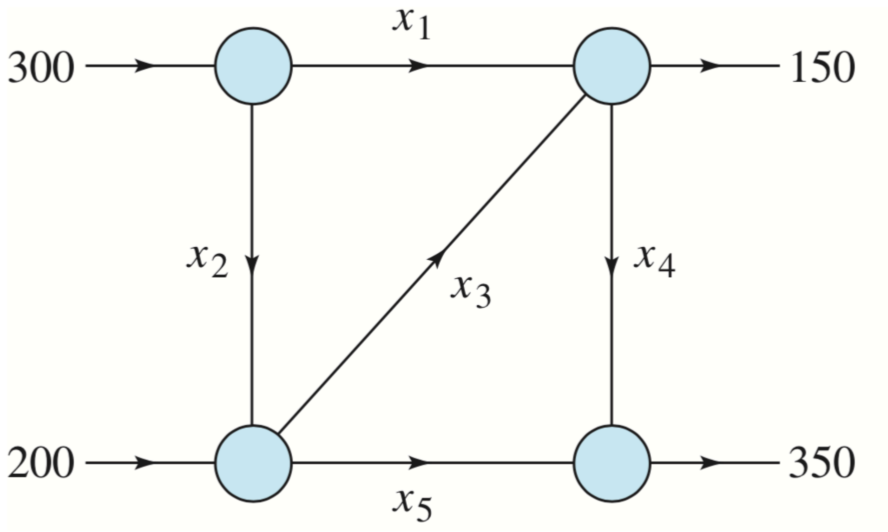
$s_i = W_{i1}x_1 + W_{i2}x_1 + \ldots + W_{im}x_m + b_i$

$$S = \begin{bmatrix} W_{11} & W_{12} & \ldots & W_{1m} & b_1 \\ W_{21} & W_{22} & \ldots & W_{2m} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ W_{n1} & W_{n2} & \ldots & W_{nm} & b_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \\ 1 \end{bmatrix} = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$$

# Linearity vs non-linearity, examples of applications, Gauss and Gauss-Jordan reduction, Echelon form, Homogeneous systems :

## Network flows

**When the EQUATIONS are GIVEN BY JUNCTIONS**




$$
\begin{aligned}
300 &= x_1 + x_2 \\
200 &= x_2 - x_3 - x_5 \\
150 &= x_1 + x_3 + x_4 \\
350 &= x_4 + x_5
\end{aligned}
$$

$$
\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & -1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}
=
\begin{bmatrix} 300 \\ 200 \\ 150 \\ 350 \end{bmatrix}
$$

**Solution in Sympy:**

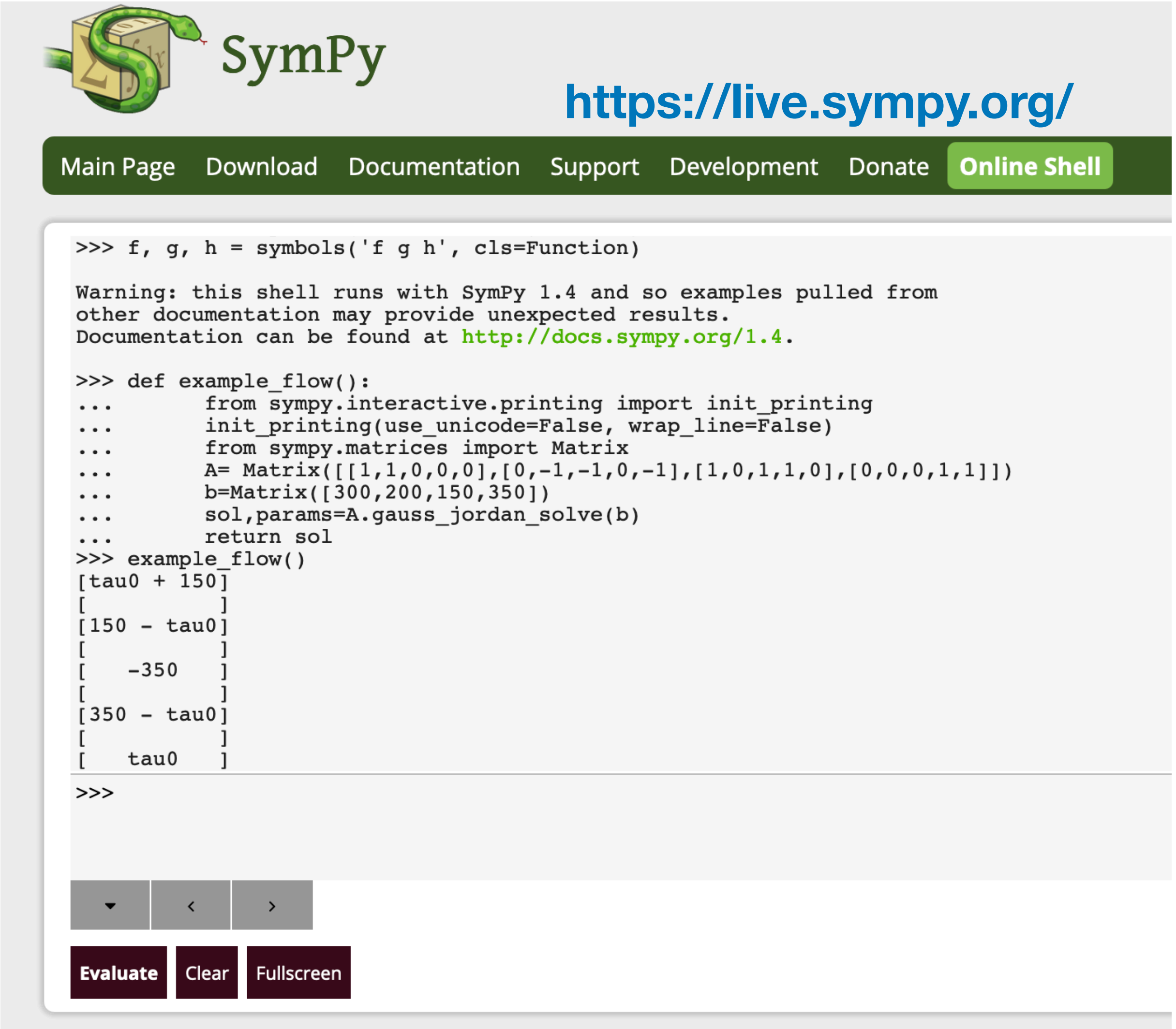
**A.gauss_jordan_solve(b)**

$$
\left( \begin{bmatrix} \tau_0 + 150 \\ 150 - \tau_0 \\ -350 \\ 350 - \tau_0 \\ \tau_0 \end{bmatrix}, \ \begin{bmatrix} \tau_0 \end{bmatrix} \right)
$$

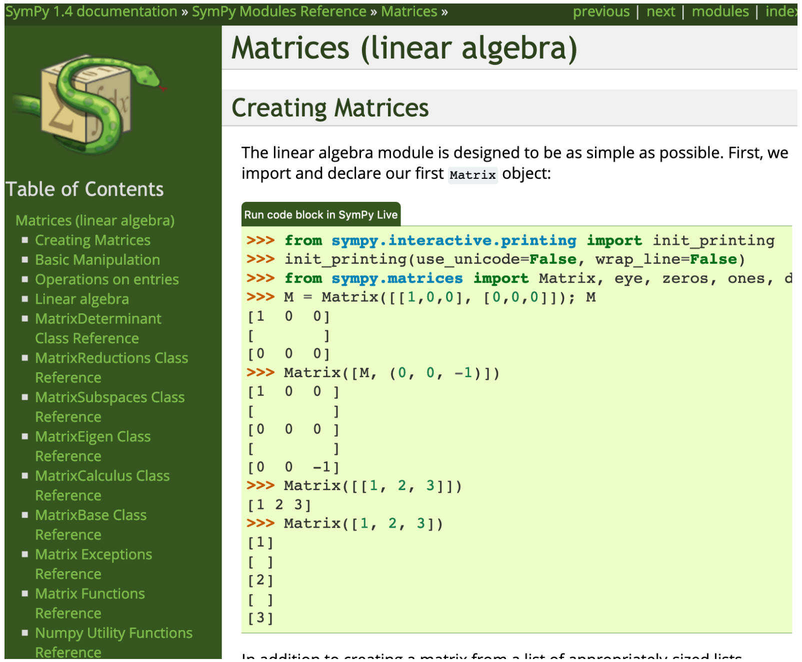
# Linearity vs non-linearity, examples of applications, Gauss and Gauss-Jordan reduction, Echelon form, Homogeneous systems :

## Solving with SymPy

**Use for solving automatically**




```
def example_flow():
    from sympy.interactive.printing import init_printing
    init_printing(use_unicode=False, wrap_line=False)
    from sympy.matrices import Matrix
    A= Matrix([[1,1,0,0,0],[0,-1,-1,0,-1],[1,0,1,1,0],[0,0,0,1,
    b=Matrix([300,200,150,350])
    sol,params=A.gauss_jordan_solve(b)
    return sol
>>>> example_flow()
```




https://docs.sympy.org/latest/modules/matrices/matrices.html#linear-algebr

# Linearity vs non-linearity, examples of applications, Gauss and Gauss-Jordan reduction, Echelon form, Homogeneous systems :

**Mixtures**

EQUATIONS = LINEAR COMBINATIONS = **TOTALS**
VARIABLES = REQUIRED QUANTITIES
COEFFICIENTS=GIVEN PROPORTIONS
ASSUMPTION=EQUALITY!

**We have three processes: A, B and C and three processors: $CPU_1$,$CPU_2$ and $CPU_3$. The total time available for $CPU_1$ is 340ms, for $CPU_2$ is 320ms and for $CPU_2$ is 140ms. However process A needs 30ms, 20ms and 10ms of each CPU, process B needs 20ms, 30ms and 10ms respectively, and C needs 20ms, 20ms and 10ms.**
**Determine, if the OS can plan these processes (ms x process)**

| | $CPU_1$ | $CPU_2$ | $CPU_3$ |
|---|---|---|---|
| **A** | 30 | 20 | 10 |
| **B** | 20 | 30 | 10 |
| **C** | 20 | 20 | 10 |
| **Totals** | **340** | **320** | **140** |

COLs

ROWs

$$\begin{bmatrix} 30 & 20 & 20 \\ 20 & 30 & 20 \\ 10 & 10 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 340 \\ 320 \\ 140 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 34 \\ 32 \\ 14 \end{bmatrix}$$

Solution: Gauss Elimination

## Linearity vs non-linearity, examples of applications, Gauss and Gauss-Jordan reduction, Echelon form, Homogeneous systems :

**Solution: Gauss Elimination**

$$\begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 34 \\ 32 \\ 14 \end{bmatrix}$$

### Operations that lead to a equivalent system

1. Interchange two rows
2. Multiply a row by a non-zero constant
3. Add a multiple of a equation to another equation

**Augmented matrix**

$$\begin{bmatrix} 3 & 2 & 2 & 34 \\ 2 & 3 & 2 & 32 \\ 1 & 1 & 1 & 14 \end{bmatrix} \xrightarrow{R_3 \leftrightarrow R_1} \begin{bmatrix} 1 & 1 & 1 & 14 \\ 2 & 3 & 2 & 32 \\ 3 & 2 & 2 & 34 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - 2R_1} \begin{bmatrix} 1 & 1 & 1 & 14 \\ 0 & 1 & 0 & 4 \\ 3 & 2 & 2 & 34 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 - 3R_1}$$

**INTERCHANGE IF FIRST ROW DOES NO HAVE A 1 PIVOT**

**ELIMINATION= MAKE ZEROS BELOW THE PIVOT**

$$\begin{bmatrix} 1 & 1 & 1 & 14 \\ 0 & 1 & 0 & 4 \\ 0 & -1 & -1 & -8 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 + R_2} \begin{bmatrix} 1 & 1 & 1 & 14 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & -1 & -4 \end{bmatrix} \xrightarrow{R_3 \leftarrow (-1)R_3} \begin{bmatrix} 1 & 1 & 1 & 14 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

**Solution=(6, 4, 4)**

# Linearity vs non-linearity, examples of applications, Gauss and Gauss-Jordan reduction, Echelon form, Homogeneous systems :

## Row-Echelon form

$$\begin{bmatrix} 1 & 1 & 1 & 14 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

**A matrix is in row-echelon form if:**

1. All rows consisting of enterily zeros occur at the bottom
2. For each row with no entirely zeros, the first non zero is 1, and it is called the leading one (pivot)
3. For two successive (non-zero) rows the leading 1 in the higher row is farther to the left that the leading 1 in the lower row

(a) $\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 1 & -3 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & -5 & 2 & -1 & 3 \\ 0 & 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(f) $\begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & -4 \end{bmatrix}$

Reduced Row-Echelon form

Gauss-Jordan Reduction

```
def rref(self, iszerofunc=_iszero, simplify=False, pivots=True, normalize_last=
    """Return reduced row-echelon form of matrix and indices of pivot vars.
```

$$\begin{bmatrix} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

## A matrix is in reduced row-echelon form

**1. It is in row echelon form**

**2. All the entries in the coefficient matrix are zero except those with leading ones.**

$$\begin{bmatrix} 1 & 1 & 1 & 14 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 4 \end{bmatrix} \xrightarrow{R_1 \leftarrow R_1 - R_2} \begin{bmatrix} 1 & 0 & 1 & 10 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 4 \end{bmatrix} \xrightarrow{R_1 \leftarrow R_1 - R_3} \begin{bmatrix} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

In rreff, the diagonal leads directly to the solution if there is only one

$$\begin{bmatrix} 1 & 1 & 1 & 14 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

Both ref and rref are equivalent! (same solutions) to the original system

$$\begin{bmatrix} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

# Linearity vs non-linearity, examples of applications, Gauss and Gauss-Jordan reduction, Echelon form, Homogeneous systems :

Number of solutions

CONSISTENT
(one or many solutions)
vs
INCONSISTENT
(no-solution)

## The rows picture

* Echelon forms explain the system in terms of rows
* A row of zeros may lead to infinite solutions
* A row with zeros in the coefficients but non-zero in the augmented column indicates non-solution (inconsistency)

$$\begin{bmatrix} 1 & 1 & 1 & 14 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

**Unique solution**

$$\begin{bmatrix} 1 & 3 & 2 & 25000 \\ 0 & 1 & -1 & -5000 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

**Infinite solutions**

$$\begin{bmatrix} 1 & 1 & 60 \\ 0 & 1 & \frac{55}{2} \\ 0 & 0 & -\frac{25}{2} \end{bmatrix}$$

**Inconsistent**

The ROW PICTURE has to do with the interpretation of solving of a system as Finding the set of points that SIMULTANEOUSLY SATISFY ALL THE EQUATIONS (Strang)

# Linearity vs non-linearity, examples of applications, Gauss and Gauss-Jordan reduction, Echelon form, Homogeneous systems :

Number of solutions

HOW MANY LINEAR COMBINATIONS?

## The cols picture

* Look at the columns of the coefficient matrix
* What linear combinations of the cols lead to the RHS?

$$\begin{bmatrix} 1 & 1 & 1 & 14 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} x_3 = \begin{bmatrix} 14 \\ 4 \\ 4 \end{bmatrix}$$

**Unique solution**

**Only $x_1=6$, $x_2=4$, $x_3=4$ leads to the RHS**

$$\begin{bmatrix} 1 & 3 & 2 & 25000 \\ 0 & 1 & -1 & -5000 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} x_3 = \begin{bmatrix} 25000 \\ -5000 \\ 0 \end{bmatrix}$$

**Infinite solutions**

**Many LC with $x_3=\lambda$ lead to the RHS**

$$\begin{bmatrix} 1 & 1 & 60 \\ 0 & 1 & \frac{55}{2} \\ 0 & 0 & -\frac{25}{2} \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} x_2 = \begin{bmatrix} 60 \\ 55/2 \\ -25/2 \end{bmatrix}$$

**NO LC leads to the RHS (we have zeros in cols)**

**Inconsistent**

# Linearity vs non-linearity, examples of applications, Gauss and Gauss-Jordan reduction, Echelon form, Homogeneous systems :

Homogeneous systems

ALWAYS HAVE SOLUTION

$$x_1 + 2x_2 - x_3 = 0$$
$$3x_1 - 3x_2 + 2x_3 = 0$$
$$-x_1 - 11x_2 + 6x_3 = 0$$

## Solutions in homogeneous systems

* These systems have only zeros in the RHS
* If the matrix of coefficients is invertible always have uniqueSolution (cols interpretation)
* Otherwise, infinite solutions

$$\begin{bmatrix} 1 & 2 & -1 & 0 \\ 3 & -3 & 2 & 0 \\ -1 & -11 & 6 & 0 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - 3R_1} \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & -9 & 5 & 0 \\ -1 & -11 & 6 & 0 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 + R_1}$$

$$\begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & -9 & 5 & 0 \\ 0 & -9 & 5 & 0 \end{bmatrix} \xrightarrow{R_2 \leftarrow -\frac{1}{9}R_2} \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & -\frac{5}{9} & 0 \\ 0 & -9 & 5 & 0 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 + 9R_2} \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & -\frac{5}{9} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

# More Examples and Exercises :

**Mixtures**

When the LINEAR COMBINATIONS OF UNKNOWNS GIVE AN EXACT QUANTITY

**We want to know how many objects of types A and B do we have if their known sum is S. Objects of type A have in turn $C_A$ pieces and objects of type B have $C_B$ pieces. It is known that $S_T = A*C_A + B*C_B$ is the sum of all the pieces required for making all the objects A+B. In addition we know that A = 2B.**
**Given S=60, $C_A$=6, $C_B$=2 and $S_T$=250, dermine A and B if possible. Otherwise explain why not.**

$$A + B = S$$
$$C_A A + C_B B = S_T$$
$$2A - B = 0$$

$$\begin{bmatrix} 1 & 1 \\ 6 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 60 \\ 250 \\ 0 \end{bmatrix}$$

$$A + B = 60$$
$$6A + 2B = 250$$
$$2A - B = 0$$

$$\frac{B}{2} + B = \frac{3}{2}B = 60 \Rightarrow B = 40,\ A = 20$$

$$6 \times 20 + 2 \times 40 = 200 \neq 250$$

$$A = \frac{B}{2}$$

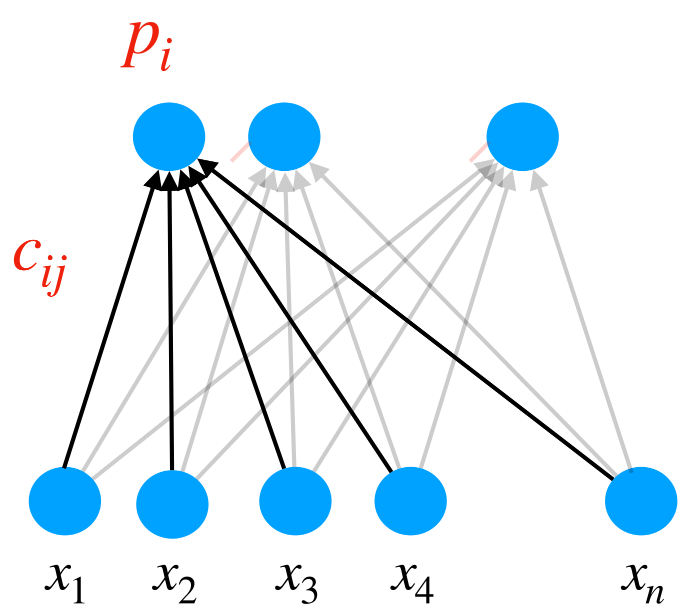
NO Solution INCONSISTENT SYSTEM

# More Examples and Exercises :

**Resource Assignment**

When the LINEAR COMBINATIONS OF UNKNOWNS must EXACTLY satisfy a set of given constraints

**Let be m machines and n workers, so that the productivity per hour of the i-th machine with the j-th worker is given by a coefficient $c_{ij}$. Each of the m machines needs to reach an exact total productivity of $p_i$ if we combine the productivities of all the workers. Determine the hours of working of each worker so that all the productivity requirements are satisfied.**




A machine can be operated by all workers

$$c_{i1}x_1 + c_{i2}x_2 + \ldots + c_{in}x_n = p_i$$

$$\begin{bmatrix} c_{11} & c_{12} & \ldots & c_{1n} \\ c_{21} & c_{22} & \ldots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \ldots & c_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_m \end{bmatrix}$$

# More Examples and Exercises :

**Resource Assignment**

When the LINEAR COMBINATIONS OF UNKNOWNS must EXACTLY satisfy a set of given constraints

$$\begin{bmatrix} 1 & 3 & 2 \\ 1 & 4 & 1 \\ 2 & 5 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 25000 \\ 20000 \\ 55000 \end{bmatrix}$$

**m=n: Unique solution?**

**Depends on the matrix of coefficients**

Gauss elimination

$$\begin{bmatrix} 1 & 3 & 2 & 25000 \\ 1 & 4 & 1 & 20000 \\ 2 & 5 & 5 & 55000 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - R_1} \begin{bmatrix} 1 & 3 & 2 & 25000 \\ 0 & 1 & -1 & -5000 \\ 2 & 5 & 5 & 55000 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 - 2R_2}$$

$$\begin{bmatrix} 1 & 3 & 2 & 25000 \\ 0 & 1 & -1 & -5000 \\ 0 & -1 & 1 & 5000 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 + R_2} \begin{bmatrix} 1 & 3 & 2 & 25000 \\ 0 & 1 & -1 & -5000 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} x_1 \geq 0 \\ x_2 = \lambda - 5000 \geq 0 \\ x_3 = \lambda \geq 0 \end{array}$$

$$x_1 = 25000 - 3(\lambda - 5000) - 2\lambda = 40000 - 5\lambda \geq 0 \quad \Rightarrow 5000 \leq x_3 \leq 8000$$

# Exercise

**Time and equations**

When the LINEAR COMBINATIONS OF UNKNOWNS are referred TO TIME COORDINATES

**Now, the linear combination of $Ax_1+Bx_2+Cx_3$ is unknown. However, 1 day ago it was decreased by 600 but modifying $x_1+1.5$, $x_2-0.5$ and $x_3+1$. Finally, 2 days ago, the linear combination increased in 350 if we had $(x_1+1.5)-1$, $(x_2-0.5)-1.5$ and $(x_3+1)+0.5$.**

**Show that: a) $x_1$, $x_2$, $x_3$ cannot be determined with the above information even when we know A, B and C, b) A,B and C, when considered as unknowns, have infinite number of solutions, c) However, if we set C=200, then A and B can be determined.**

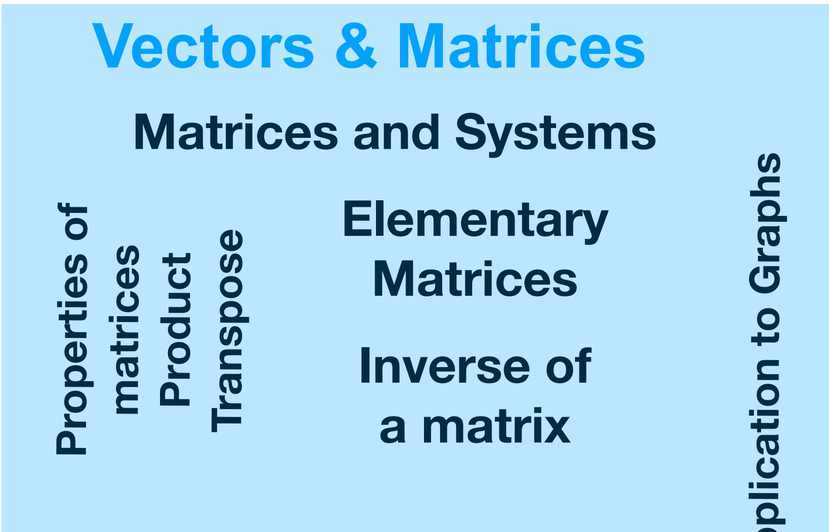
SOLVE THIS SYSTEM WITH SymPy. Before solving, apply the rref function and show the Reduced Echelon form and the pivots

```
def example_rref():
    from sympy.interactive.printing import init_printing
    init_printing(use_unicode=False, wrap_line=False)
    from sympy.matrices import Matrix
    Ab= Matrix([[1,1,0,0,0,300],[0,-1,-1,0,-1,200],[1,0,1,1,0,150],[0,0,0,1,1,350]])
    r,pivots = Ab.rref()
    return r,pivots
```

# 2. Vectors and Matrices

**Linearity vs non-linearity, examples of applications, Gauss and Gauss-Jordan reduction, Echelon form, Applications to graphs:**

Francisco Escolano



## Linearity vs non-linearity, examples of applications, Gauss and Gauss-Jordan reduction, Echelon form, Homogeneous systems :

**TO BE CONTINUED…**